

# Finite-Time Stabilization of Switched Systems with Unstable Modes

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**Abstract**—In this paper, we study finite-time stability and stabilization of switched systems in the presence of unstable modes. In contrast to asymptotic or exponential stability where the system trajectories reach the equilibrium point as time tends to infinity, the notion of finite-time stability requires the trajectories to reach the equilibrium within a finite amount of time. We show that even if the value of the Lyapunov function increases in between two switches, i.e., if there are unstable modes in the system, finite-time stability can still be guaranteed if the finite-time convergent mode is active for a long enough cumulative time duration. Then, we present a method for the synthesis of a finite-time stabilizing switching signal. As a case study, we design a finite-time stable, output-feedback controller for a linear switched system, in which only one of the modes is both controllable and observable.

## I. INTRODUCTION

The stability of switched systems has been investigated by many researchers in the past, typically using either a *common* Lyapunov function or *multiple* Lyapunov functions. In [1], the necessity and sufficiency of the existence of a common Lyapunov function for all the subsystems of a switched system for asymptotic stability under arbitrary switching are discussed. The authors in [2] study linear switched systems with dwell time using a common quadratic control Lyapunov function (CQLF) and state-space partitioning. In the review article [3], the authors study the stability of switched linear systems and linear differential inclusions. They present sufficient conditions for the existence of CQLFs and discuss converse Lyapunov results for switched systems.

In [4], the author introduces the concept of multiple Lyapunov functions to analyze the stability of switched systems; since then, a lot of work has been done on the stability of switched systems using multiple Lyapunov functions [5], [6]. [5] relaxes the non-increasing condition on the Lyapunov functions by introducing generalized Lyapunov functions, and provides necessary and sufficient conditions for the stability of switched systems under arbitrary switching.

In contrast to asymptotic stability (AS), which pertains to the convergence of the trajectories to the equilibrium as time tends to infinity, finite-time stability (FTS)<sup>1</sup> is a concept that requires the convergence of the trajectories to the equilibrium in finite time. FTS is a well-studied concept, motivated in part from a practical viewpoint due to properties such as convergence in a finite time, and robustness with respect to disturbances. In the seminal work [7], the authors introduce necessary and sufficient conditions in terms of Lyapunov functions for dynamical systems to exhibit FTS, with a focus on continuous-time autonomous systems.

FTS of switched systems has gained popularity in the last few years. The authors in [8] consider the problem of

designing a controller for a linear switched system under delay and external disturbance with finite- and fixed-time convergence. In [9], the authors design a hybrid observer and show finite-time convergence in the presence of unknown, constant bias. The authors in [10] design an FTS state observer for switched systems via a sliding-mode technique. In [11], the authors introduce the concept of a *locally* homogeneous system and show FTS of switched systems with uniformly bounded uncertainties. More recently, [12] studies FTS of homogeneous switched systems by introducing the concept of hybrid homogeneous degree and relating negative homogeneity with FTS. In [13], the authors consider systems in strict-feedback form with positive powers and design a controller as well as a switching law so that the closed-loop system is FTS. In [14], the authors design an FTS observer for switched systems with unknown inputs. They assume that each linear subsystem is *strongly* observable and that the first switching occurs after an *a priori* known time.

In this paper, we consider a general class of switched systems and develop sufficient conditions for FTS of the origin of the switched system in terms of multiple Lyapunov functions. Compared to earlier work in [11], [12], [13], we do not assume that the subsystems are homogeneous or in strict-feedback form, and present conditions in terms of multiple Lyapunov functions for ensuring FTS of the origin. *To the best of authors' knowledge, this is the first work considering FTS of a general class of switched systems using multiple Lyapunov functions.* The main contributions are as follows.

**FTS of switched systems:** We present sufficient conditions for FTS using multiple Lyapunov functions allowing the Lyapunov functions to have bounded increment. More specifically, we relax the requirement in [15] that the Lyapunov function is strictly decreasing during the time intervals in which the corresponding subsystem is active. Instead, we allow the Lyapunov functions to increase during such time intervals and only require that these increments are bounded, thus allowing unstable modes while still guaranteeing FTS.

**A switching signal for FTS:** We use the proposed multiple Lyapunov function framework to design a switching signal so that the origin of the resulting switched system is FTS. As an application, we study the problem of a switched linear control system to stabilize the origin using output feedback for the case when only one of the subsystems (or modes) is controllable and observable. We design an FTS observer to reconstruct the state vector in a finite time, and an FTS controller to render the origin of the resulting closed-loop switched system finite-time stable.

The motivation of studying FTS using multiple Lyapunov functions comes from applications where the switching law is not under the user's control authority, or where keeping the FTS mode active for a long period leads to undesirable behavior. As an example, consider a spacecraft that tracks the desired trajectory, with the onboard communication and the controller module requiring a certain minimum energy threshold to function. The charge level of the spacecraft

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<sup>1</sup>With slight abuse of notation, we use FTS to denote the phrase "finite-time stability" or "finite-time stable", depending on the context.

battery can be modeled as a switched system, where being in the path of sunlight would be an FTS mode, leading to an increase in the charging level, and tracking the desired trajectory an unstable mode since it depletes the charge. Now, keeping the FTS mode active for a long duration might lead to the spacecraft losing track of its desired trajectory, and thus, the switching signal between the two modes cannot be designed arbitrarily. At the same time, FTS is desired so that the spacecraft can activate its communication module for crucial communications with the ground station and/or the control module to compute inputs for the next part of the journey. Thus, for the applications where the FTS mode cannot be kept active at all times, or the switching signal is not under the user's control, it is essential to study FTS under switching laws that allow the FTS mode to become inactive, and unstable modes to become active.

## II. PRELIMINARIES

We denote by  $\|\cdot\|$  the Euclidean norm of vector  $(\cdot)$ ,  $|\cdot|$  the absolute value if  $(\cdot)$  is scalar and the length if  $(\cdot)$  is a time interval. The set of non-negative reals is denoted by  $\mathbb{R}_+$ , set of non-negative integers by  $\mathbb{Z}_+$  and set of positive integers by  $\mathbb{N}$ . We denote by  $\text{int}(S)$  the interior of the set  $S$ , and by  $t^-$  and  $t^+$  the time just before and after the time instant  $t$ , respectively.

**Definition 1.** A continuous function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a class- $\mathcal{K}$  function if it is strictly increasing with  $\alpha(0) = 0$ . It is class- $\mathcal{K}_\infty$  if, in addition,  $\lim_{r \rightarrow \infty} \alpha(r) = \infty$ .

Consider the system:

$$\dot{y} = f(y), \quad (1)$$

where  $y \in \mathbb{R}^n$ ,  $f : D \rightarrow \mathbb{R}^n$  is continuous on an open neighborhood  $D \subseteq \mathbb{R}^n$  of the origin and  $f(0) = 0$ . Assume that the solution of (1) exists and is unique. The origin is said to be an FTS equilibrium of (1) if it is Lyapunov stable and finite-time convergent, i.e., for all  $y(0) \in \mathcal{N} \setminus \{0\}$ , where  $\mathcal{N}$  is some open neighborhood of the origin,  $\lim_{t \rightarrow T} y(t) = 0$ , where  $T = T(y(0)) < \infty$  [7]. The authors also presented Lyapunov conditions for FTS of the origin of (1). The following result is adapted from [7]:

**Lemma 1.** Suppose there exists a continuously differentiable function  $V : D \rightarrow \mathbb{R}$  such that the following holds: (i)  $V$  is positive definite. (ii) There exist real numbers  $c > 0$  and  $\alpha \in (0, 1)$ , and an open neighborhood  $\mathcal{V} \subseteq D$  of the origin such that  $\dot{V}(y) \leq -cV(y)^\alpha$ ,  $y \in \mathcal{V} \setminus \{0\}$ . Then the origin is an FTS equilibrium for (1).

Consider the switched system

$$\dot{x}(t) = f_{\sigma(t,x)}(x(t)), \quad x(t_0) = x_0, \quad (2)$$

where  $x \in \mathbb{R}^n$  is the system state,  $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \Sigma$  is a piecewise constant switching signal that can depend both on state and time,  $\Sigma \triangleq \{1, 2, \dots, N\}$  with  $N < \infty$ , and  $f_{\sigma(\cdot, \cdot)} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the system vector field describing the active subsystem (called thereafter mode) under  $\sigma(\cdot, \cdot)$ . The switching signal  $\sigma$  is assumed to be piecewise continuous in  $x$  and right-continuous in  $t$ . Let  $\mathcal{I} = [0, \tau)$ , with  $\tau > 0$  be the domain of definition of the solutions of (2). A solution  $x(\cdot)$  of (2) is an absolutely continuous function that satisfies  $\dot{x}(t) = f_k(x(t))$  for almost all  $t \in \mathcal{I}$  where  $k = \sigma(t, x(t)) \in \Sigma$ . The solution is maximal if  $\mathcal{I}$  cannot be extended, and is

complete if  $\tau = \infty$ . Denote by  $T_{i_k} = [t_{i_k}, t_{i_k+1}) \subset \mathbb{R}_+$ , with  $t_{i_k+1} \geq t_{i_k} \geq t_0$ , the interval in which the flow  $f_i$  is active for the  $k$ -th time for  $i \in \Sigma$  and  $k \in \mathbb{N}$ . The solution of (2) is Zeno if there are infinite number of switches in the signal  $\sigma$  within a finite amount of time. We make the following assumption for (2).

**Assumption 1.** The solution of (2) exists, is non-Zeno and is complete. In addition, there is a non-zero dwell-time for the FTS mode  $F \in \Sigma$ , i.e.,  $|T_{F_k}| = t_{F_{k+1}} - t_{F_k} \geq t_d$  for all  $k \in \mathbb{N}$ , where  $t_d > 0$  is a positive constant.

We state the following lemma before we proceed to the main results.

**Lemma 2.** Let  $a_i \geq b_i \geq 0$  for all  $i \in \{1, 2, \dots, K\}$  for some  $K \in \mathbb{N}$ . Then, for any  $0 < r < 1$ , we have

$$\sum_{i=1}^K (a_i^r - b_i^r) \leq \sum_{i=1}^K (a_i - b_i)^r. \quad (3)$$

The proof can be completed using [16, Lemma 3.3].

## III. MAIN RESULTS

### A. FTS of Switched Systems

In this section, we present our main result on sufficient conditions for FTS of the origin of (2) in terms of multiple Lyapunov functions. Let  $\{i^0, i^1, \dots, i^p, \dots\}$  be the sequence of modes that are active during the intervals  $[t_0, t_1)$ ,  $[t_1, t_2)$ ,  $\dots$ ,  $[t_p, t_{p+1})$ ,  $\dots$ , respectively, for  $i^p \in \Sigma$ ,  $p \in \mathbb{Z}_+$ .

**Theorem 1.** If there exist functions  $V_i$  for each  $i \in \Sigma$ , and a switching signal  $\sigma$  such that the following hold:

(i) There exists  $\alpha_1 \in \mathcal{K}$ , such that

$$\sum_{k=0}^p \left( V_{i^{k+1}}(x(t_{k+1})) - V_{i^k}(x(t_{k+1})) \right) \leq \alpha_1(\|x_0\|), \quad (4)$$

holds for all  $p \in \mathbb{Z}_+$ ;

(ii) There exists  $\alpha_2 \in \mathcal{K}$  such that

$$\sum_{k=0}^p \left( V_{i^k}(x(t_{k+1})) - V_{i^k}(x(t_k)) \right) \leq \alpha_2(\|x_0\|), \quad (5)$$

holds for all  $p \in \mathbb{Z}_+$ ;

(iii) There exist a positive definite, continuously differentiable Lyapunov function  $V_F$  and constants  $c > 0$ ,  $0 < \beta < 1$  such that

$$\dot{V}_F(x(t)) \leq -c(V_F(x(t)))^\beta \quad \forall t \in \bigcup_k [t_{F_k}, t_{F_{k+1}}); \quad (6)$$

(iv) The accumulated duration  $|T_F| \triangleq \sum_k |T_{F_k}|$  corresponding to the period of time during which the mode  $F$  is active, satisfies

$$|T_F| = \gamma(\|x_0\|) \triangleq \frac{(\alpha(\|x_0\|))^{1-\beta}}{c(1-\beta)} + \frac{M^{-\beta}(\bar{\alpha}(\|x_0\|))^{1-\beta}}{c(1-\beta)},$$

where  $\alpha = \alpha_0 + \alpha_1 + \alpha_2$ ,  $\bar{\alpha} = M(\alpha_1 + \alpha_2)$  and  $\alpha_0 \in \mathcal{K}$ , and  $M < \infty$  is the number of times the mode  $F$  is activated,

then, the origin of (2) is FTS with respect to the switching signal  $\sigma$ . Moreover, if all the conditions hold globally, the functions  $V_i$  are radially unbounded for all  $i \in \Sigma$ , and  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ , then the origin of (2) is globally FTS.

We provide an intuitive explanation of the conditions of Theorem 1 (see Figure 1).

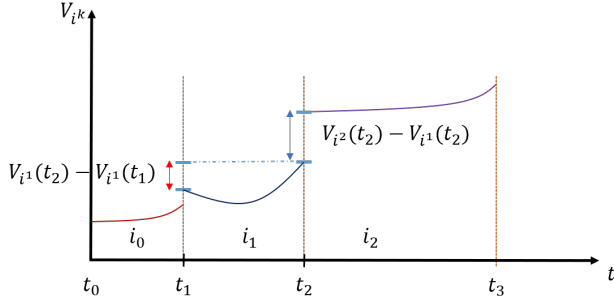


Fig. 1. Conditions (i) and (ii) Theorem 1 regarding the allowable changes in the values of the Lyapunov functions. The increments shown by blue and red double-arrows pertain to condition (i) and (ii), respectively.

- Condition (i): the cumulative value of the differences between the consecutive Lyapunov functions at the switching instants  $t_i$  is bounded by a class- $\mathcal{K}$  function.
- Condition (ii): the cumulative increment in the values of the individual Lyapunov functions when the respective modes are active is bounded by a class- $\mathcal{K}$  function.<sup>2</sup>
- Condition (iii): there exists an FTS mode  $F \in \Sigma$  and a Lyapunov function  $V_F$  satisfying (6) for  $\dot{x}(t) = f_F(x(t))$  on  $[t_{F_k}, t_{F_{k+1}}]$  for all  $k \in \mathbb{Z}_+$ .
- Condition (iv): the FTS mode  $F$  is active for a sufficiently long cumulative time  $\gamma(\|x_0\|)$ .

Now we provide the proof of Theorem 1.

*Proof.* First we prove the stability of the origin under conditions (i)-(ii). Let  $x_0 \in D$ , where  $D$  is an open neighborhood of the origin. For all  $p \in \mathbb{Z}_+$ , we have that

$$\begin{aligned} V_{i^p}(x(t_p)) &= V_{i^0}(x(t_0)) + \sum_{k=1}^p \left( V_{i^k}(x(t_k)) - V_{i^{k-1}}(x(t_k)) \right) \\ &\quad + \sum_{k=0}^{p-1} \left( V_{i^k}(x(t_{k+1})) - V_{i^k}(x(t_k)) \right) \\ &\stackrel{(4),(5)}{\leq} \alpha_0(\|x_0\|) + \alpha_1(\|x_0\|) + \alpha_2(\|x_0\|) = \alpha(\|x_0\|) \end{aligned}$$

where  $\alpha = \alpha_0 + \alpha_1 + \alpha_2$  with  $\alpha_0(r) = \max_{i \in \Sigma_f, \|x\| \leq r} V_i(x)$ .

Thus, we have:

$$V_{i^p}(x(t_p)) \leq \alpha(\|x_0\|), \quad (7)$$

for all  $p \in \mathbb{Z}_+$ . Let  $d_i(c) = \{x \mid V_i(x) \leq c\}$  denote the  $c$  sub-level set of the Lyapunov function  $V_i$ ,  $i \in \Sigma_f$ , and  $B_\rho = \{x \mid \|x\| \leq \rho\}$  denote a ball centered at the origin with radius  $\rho \in \mathbb{R}_+$ . Define  $r(c) = \inf\{\rho \geq 0 \mid d_i(c) \subset B_\rho\}$  as the radius of the smallest ball centered at the origin that encloses the  $c$  sub-level sets  $d_i(c)$ , for all  $i \in \Sigma_f$ . Since the functions  $V_i$  are positive definite, the sub-level sets  $d_i(c)$  are bounded for small  $c > 0$ , and hence, the function  $r$  is invertible. The inverse function  $c_\epsilon = r^{-1}(\epsilon)$  maps the radius  $\epsilon > 0$  to the value  $c_\epsilon$  such that the sub-level sets  $d_i(c_\epsilon)$  are contained in  $B_\epsilon$  for all  $i \in \Sigma_f$ . For any given  $\epsilon > 0$ , choose  $\delta = \alpha^{-1}(r^{-1}(\epsilon)) > 0$  so that (7) implies that for  $\|x_0\| \leq \delta$ , we have  $\|x(t_p)\| \leq \epsilon$  for all  $p \in \mathbb{Z}_+$ , i.e., the origin is Lyapunov stable (LS).

Next, we prove FTS of the origin when conditions (iii)-(iv) also hold. From (7), we have that

<sup>2</sup>Note that some authors use the time derivative condition, i.e.,  $\dot{V}_i \leq \lambda V_i$  with  $\lambda > 0$ , in place of condition (ii), to allow growth of  $V_i$ , hence, requiring the function to be continuously differentiable (see, e.g., [17]). Our condition allows the use of non-differentiable Lyapunov functions.

$$V_F(x(t_{F_i})) \leq \alpha(\|x_0\|), \quad (8)$$

for all  $i \in \mathbb{N}$ . Let  $M \in \mathbb{N}$  denote the total number of times the mode  $F$  is activated. We integrate (6) to obtain  $|T_{F_k}| \leq \frac{V_{F_k}^{1-\beta}}{c(1-\beta)} - \frac{V_{F_{k+1}}^{1-\beta}}{c(1-\beta)}$ . Thus, for any  $M \in \mathbb{N}$ , we have:

$$\begin{aligned} \sum_{k=1}^M |T_{F_k}| &\leq \sum_{k=1}^M \left( \frac{V_{F_k}^{1-\beta}}{c(1-\beta)} - \frac{V_{F_{k+1}}^{1-\beta}}{c(1-\beta)} \right) \\ &= \frac{V_{F_1}^{1-\beta}}{c(1-\beta)} + \sum_{i=1}^{M-1} \frac{V_{F_{i+1}}^{1-\beta} - V_{F_i}^{1-\beta}}{c(1-\beta)} - \frac{V_{F_{M+1}}^{1-\beta}}{c(1-\beta)}. \end{aligned}$$

Using (8), we obtain that  $\frac{V_{F_1}^{1-\beta}}{c(1-\beta)} \leq \frac{(\alpha(\|x_0\|))^{1-\beta}}{c(1-\beta)}$ . Define  $\gamma_1(\|x_0\|) \triangleq \frac{(\alpha(\|x_0\|))^{1-\beta}}{c(1-\beta)}$  and note that  $\gamma_1 \in \mathcal{K}$ . Now, let  $F_s = \{q_1, q_2, \dots, q_k\}, 0 \leq q_l \leq M$ , be the set of indices such that  $V_{F_{i+1}} \geq V_{F_i}$  for  $i \in F_s$ . We know that for  $a \geq b \geq 0$ ,  $a^r \geq b^r$  for any  $r > 0$ . Hence, we have that

$$\sum_{i=1}^{M-1} \frac{V_{F_{i+1}}^{1-\beta} - V_{F_i}^{1-\beta}}{c(1-\beta)} \leq \sum_{i \in F_s} \frac{V_{F_{i+1}}^{1-\beta} - V_{F_i}^{1-\beta}}{c(1-\beta)} \quad (9)$$

Using Lemma 2, we obtain that

$$\sum_{i \in F_s} \frac{V_{F_{i+1}}^{1-\beta} - V_{F_i}^{1-\beta}}{c(1-\beta)} \leq \sum_{i \in F_s} \frac{(V_{F_{i+1}} - V_{F_i})^{1-\beta}}{c(1-\beta)}. \quad (10)$$

From the analysis in the first part of the proof, we know that

$$\begin{aligned} V_{F_{i+1}} - V_{F_i} &= \sum_{k=l_1}^{l_2} \left( V_{i^k}(x(t_k)) - V_{i^{k-1}}(x(t_k)) \right) \\ &\quad + \sum_{k=l_1}^{l_2-1} \left( V_{i^k}(x(t_{k+1})) - V_{i^k}(x(t_k)) \right) \\ &\leq \alpha_1 + \alpha_2, \end{aligned}$$

where  $l_1, l_2$  are such that  $t_{l_1}$  denotes the time when mode  $F$  becomes deactivated for the  $i$ -th time and  $t_{l_2}$  denotes the time when the mode  $F$  is activated for  $(i+1)$ -th time. Define  $\bar{\alpha} = M(\alpha_1 + \alpha_2)$  so that we have

$$\sum_{i \in F_s} (V_{F_{i+1}} - V_{F_i}) \leq \bar{\alpha}(\|x_0\|). \quad (11)$$

Hence, we have that

$$\begin{aligned} \sum_{i=1}^{M-1} \frac{V_{F_{i+1}}^{1-\beta} - V_{F_i}^{1-\beta}}{c(1-\beta)} &\stackrel{(10)}{\leq} \frac{\sum_{i \in F_s} (V_{F_{i+1}} - V_{F_i})^{1-\beta}}{c(1-\beta)} \\ &\leq \frac{M^{-\beta} \left( \sum_{i \in F_s} V_{F_{i+1}} - \bar{V}_{F_{i+1}} \right)^{1-\beta}}{c(1-\beta)} \\ &\stackrel{(11)}{\leq} \frac{M^{-\beta} (\bar{\alpha}(\|x_0\|))^{1-\beta}}{c(1-\beta)}, \quad (12) \end{aligned}$$

where the second inequality follows from [16, Lemma 3.4].

Define  $\gamma(\|x_0\|) \triangleq \gamma_1(\|x_0\|) + \frac{M^{-\beta} (\bar{\alpha}(\|x_0\|))^{1-\beta}}{c(1-\beta)}$  and  $|T_F| = \sum_{k=1}^M |T_{F_k}|$  so that we obtain:

$$|T_F| + \frac{V_{F_{M+1}}^{1-\beta}}{c(1-\beta)} \leq \frac{V_{F_1}^{1-\beta}}{c(1-\beta)} + \sum_{i=1}^{M-1} \frac{V_{F_{i+1}}^{1-\beta} - V_{F_i}^{1-\beta}}{c(1-\beta)} \leq \gamma(\|x_0\|).$$

Clearly,  $\gamma \in \mathcal{K}$ . Now, with  $|T_F| = \gamma(\|x_0\|)$ , we obtain  $|T_F| + \frac{V_{F_{M+1}}^{1-\beta}}{c(1-\beta)} \leq \gamma(\|x_0\|) = |T_F|$ , which implies that  $\frac{V_{F_{M+1}}^{1-\beta}}{c(1-\beta)} \leq 0$ . However,  $V_F \geq 0$ , which further implies that  $V_{F_{M+1}} = 0$ . Hence, if mode  $F$  is active for the accumulated time  $|T_F| = \gamma(\|x_0\|)$ , the value of the function  $V_F$  converges to 0 as  $t \rightarrow t_{F_{M+1}}$ , and thus, the origin of (2) is FTS. Note that under Assumption 1, it follows that  $M \leq \frac{\gamma(\|x_0\|)}{t_d} < \infty$ .

Finally, if all the conditions (i)-(iv) hold globally and the functions  $V_i$  are radially unbounded, we have that  $\alpha_0$  is also radially unbounded and  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ . Thus, we have that  $\alpha(\|x_0\|) < \infty$  and  $\bar{\alpha}(\|x_0\|) < \infty$  for all  $\|x_0\| < \infty$ , and hence,  $\gamma(\|x_0\|) < \infty$  for all  $\|x_0\| < \infty$ , which implies global FTS of the origin.  $\square$

### B. Finite-Time Stabilizing Switching Signal

In this section, we present a method of designing a switching signal based upon Theorem 1, so that the origin of the switched system is FTS. The approach is inspired from [5] where a method of designing an asymptotically stabilizing switching signal is presented. Suppose there exist continuous functions  $\mu_{ij} : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying:

$$\begin{aligned} \mu_{ij}(0) = 0, \quad \mu_{ii}(x) = 0 \quad \forall x, \\ \mu_{ij}(x) + \mu_{jk}(x) \leq \min\{0, \mu_{ik}(x)\}, \quad \forall x \end{aligned} \quad (13)$$

for all  $i, j, k \in \Sigma$ . Define the following sets:

$$\begin{aligned} \Omega_i &= \{x \mid V_i(x) - V_j(x) + \mu_{ij}(x) \leq 0, \forall j \in \Sigma\}, \\ \Omega_{ij} &= \{x \mid V_i(x) - V_j(x) + \mu_{ij}(x) = 0, i \neq j\}, \end{aligned} \quad (14)$$

where  $V_i : \mathbb{R}^n \rightarrow \mathbb{R}$  for each  $i \in \Sigma$ .

Now we are ready to define the switching signal. Let  $\sigma(t_0, x(t_0)) = i$  and  $i, j \in \Sigma$  be any arbitrary modes. For all times  $t \geq t_0$ , define the switching signal as:

$$\sigma(t, x) = \begin{cases} i, & \sigma(t^-, x(t^-)) = i, x(t^-) \in \text{int}(\Omega_i); \\ j, & \sigma(t^-, x(t^-)) = i, x(t^-) \in \Omega_{ij}; \end{cases} \quad (15)$$

We now state the following result.

**Theorem 2.** *Assume that the solution of (2) under  $\sigma$  in (15) satisfies Assumption 1. Let  $V_i$  and  $\mu_{ij}$  satisfy (13). Assume that the following hold:*

- (I) *There exist continuous functions  $\beta_{ij} : \mathbb{R}^n \rightarrow \mathbb{R}$  for  $i, j \in \Sigma$  such that  $\beta_{ij}(x) \leq 0$  for all  $x \in \mathbb{R}^n$  and*

$$\frac{\partial V_i}{\partial x} f_i(x) + \sum_{j=1}^{N_f} \beta_{ij}(x)(V_i(x) - V_j(x) + \mu_{ij}(x)) \leq 0 \quad (16)$$

*holds for all  $i \in \Sigma$ , for all  $x \in \mathbb{R}^n$ ;*

- (II) *There exists a finite-time stable mode  $F \in \Sigma$  satisfying conditions (iii) and (iv) of Theorem 1;*

- (III) *The functions  $\mu_{ij}$  are continuously differentiable and satisfy*

$$\frac{\partial \mu_{ij}}{\partial x} f_i \leq 0, \quad i, j = 1, 2, \dots, N. \quad (17)$$

- (IV) *No sliding mode occurs at any switching surface.*

*Then, the origin of (2) under  $\sigma$  in (15) is FTS.*

*Proof.* We show that all the conditions of Theorem 1 are satisfied to establish FTS of the origin for (2), when the switching signal is defined as per (15). As per the analysis

in [5, Theorem 3.18], we obtain that the conditions (i)-(ii) of Theorem 1 are satisfied with

$$\alpha_1(r) = \max_{\|x\| \leq r, i, j \in \Sigma_f} |\mu_{ij}(x)|, \quad (18)$$

$$\alpha_2(r) = 0, \quad (19)$$

for any  $r \geq 0$ . From (II), we obtain that conditions (iii) and (iv) of Theorem 1 hold as well. Thus, all the conditions of the Theorem 1 are satisfied. Hence, we obtain that the origin of (2) with switching signal defined as per (15) is FTS.  $\square$

**Remark 1.** *Note that an arbitrarily defined switching signal  $\sigma$  does not necessarily satisfy the conditions of Theorem 1, particularly condition (iv), where the mode  $F$  is required to be active for  $T_F(x_0)$  time duration. However, for any given initial condition  $x_0$ , the switching signal can be defined as per (15) to render the origin of (2) FTS. Note that there is no difference in the switching signal defined in (15) and the one in [5]. This observation re-emphasizes the fact that a system whose origin is stable can be made FTS by ensuring that the cumulative activation time requirement is satisfied for an FTS mode.*

**A note on construction of functions  $\mu_{ij}, V_i$ :** For a class of switched systems consisting of  $N-1$  linear modes and one FTS mode  $F$ , one can follow a design procedure similar to [5, Remark 3.21] to construct the functions  $\mu_{ij}$ , as well as the Lyapunov functions  $V_i$ , for all  $i \neq F$ . The design procedure includes choosing quadratic functions  $\mu_{ij} = x^T P_{ij} x$  and  $V_i = x^T R_i x$  with  $R_i$  as positive definite matrices, and using the conditions (13) and (17) along with the conditions of Theorem 1, to formulate a linear matrix inequality (LMI) based optimization problem. For system consisting of polynomial right-hand side dynamics  $f_i$ , one can formulate a sum-of-square (SOS) problem to find polynomial functions  $V_i, \mu_{ij}$  and  $\beta_{ij}$  by posing (13), (16) and (17) inequalities as SOS constraints (see e.g., [18] for methods of solving SOS problems). The ‘‘min-switching’’ law as described in [19], can be defined by setting the functions  $\mu_{ij} = 0$ , which would imply that the Lyapunov functions should be non-increasing at the switching instants. Our conditions on the lines of the generalization of min-switching law, as presented in [5], overcome this limitation and allow the Lyapunov functions to increase at the switching instants.

### C. FTS output-feedback for Switched Linear Systems

In this section, we consider a switched linear system with  $N$  modes such that only one mode is observable and controllable, and design an output-feedback controller to stabilize the system trajectories at the origin in a finite time. Consider the system:

$$\begin{aligned} \dot{x} &= A_{\sigma(t,x)} x + B_{\sigma(t,x)} u, \\ y &= C_{\sigma(t,x)} x, \end{aligned} \quad (20)$$

where  $x \in \mathbb{R}^n, u \in \mathbb{R}, y \in \mathbb{R}$  are the system states, and input and output of the system, respectively, with  $A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times 1}$  and  $C_i \in \mathbb{R}^{1 \times n}$ . The switching signal  $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \Sigma \triangleq \{1, 2, \dots, N\}$  is a piecewise constant function that is piecewise continuous in  $x$  and right-continuous in  $t$ . We make the following assumption:

**Assumption 2.** *There exists a mode  $\sigma_0 \in \Sigma$  such that  $(A_{\sigma_0}, B_{\sigma_0})$  is controllable and  $(A_{\sigma_0}, C_{\sigma_0})$  is observable.*

Without loss of generality, one can assume that the pair  $(A_{\sigma_0}, C_{\sigma_0})$  is in the controllable canonical form and  $(A_{\sigma_0}, C_{\sigma_0})$  is in the observable canonical form, i.e.,  $A_{\sigma_0} = \begin{bmatrix} 0_n & I_{n-1} \\ 1 & 0 \end{bmatrix}$ ,  $B_{\sigma_0} = [0 \ 0 \ 0 \ \cdots \ 0 \ 1]^T$  and  $C_{\sigma_0} = [1 \ 0 \ 0 \ \cdots \ 0 \ 0]$ , where  $I_{n-1} \in \mathbb{R}^{(n-1) \times (n-1)}$  is an identity matrix and  $0_k = [0 \ 0 \ \cdots \ 0 \ 0]^T \in \mathbb{R}^{k \times 1}$ .

The objective is to design an output feedback for (20) so that the closed loop trajectories  $x(\cdot)$  reach the origin in a finite time. To this end, we first design an FTS observer, and use the estimated states  $\hat{x}$  to design the control input  $u$ . The form of the observer is:

$$\dot{\hat{x}} = A_\sigma \hat{x} + g_\sigma(C_\sigma x - C_\sigma \hat{x}) + B_\sigma u. \quad (21)$$

Following [20, Theorem 10], we define the function  $g: \mathbb{R} \rightarrow \mathbb{R}^n$  as  $g_i(y) = l_i \text{sign}(y)|y|^{\alpha_i}$ ,  $i = 1, 2, \dots, n$ , where  $l_i$  are such that the matrix  $\bar{A}$  defined as  $\bar{A} = \begin{bmatrix} -\bar{l} & I_{n-1} \\ 0_{n-1} \end{bmatrix}$  where  $\bar{l} = [l_1 \ l_2 \ \cdots \ l_n]^T$  is Hurwitz, and the exponents  $\alpha_i$  are chosen as  $\alpha_i = i\alpha - (i-1)$  for  $1 < i \leq n$ , where  $1 - \frac{n-1}{n} < \alpha < 1$ . Define the function  $g_\sigma$  as:

$$g_\sigma(y) = \begin{cases} g(y), & \sigma(t) = \sigma_0; \\ 0, & \sigma(t) \neq \sigma_0; \end{cases} \quad (22)$$

Let the observation error be  $e = x - \hat{x}$ , with  $e_i = x_i - \hat{x}_i$  for  $i = 1, 2, \dots, N$ . Its time derivative reads:

$$\dot{e} = A_\sigma e - g_\sigma(C_\sigma e). \quad (23)$$

Next, we design a feedback  $u = u(\hat{x})$  so that the origin is FTS for the closed-loop trajectories of (20). Inspired from control input defined in [21, Proposition 8.1], we define the control input as

$$u(\hat{x}) = \begin{cases} -\sum_{i=1}^n k_i \text{sign}(\hat{x}_i)|\hat{x}_i|^{\beta_i}, & \sigma(t) = \sigma_0; \\ 0, & \sigma(t) \neq \sigma_0; \end{cases} \quad (24)$$

where  $\beta_{j-1} = \frac{\beta_j \beta_{j+1}}{2\beta_{j+1} - \beta_j}$  with  $\beta_{n+1} = 1$  and  $0 < \beta_n = \beta < 1$ , and  $k_i$  are such that the polynomial  $s^n + k_n s^{n-1} + \cdots + k_2 s + k_1$  is Hurwitz. We now state the following result.

**Theorem 3.** *Let the switching signal  $\sigma$  for (20) be given by (15) with  $F = \sigma_0$  and assume that the closed-loop solutions of (20) under the control input (24) and switching signal (15) satisfy Assumption 1. Assume that there exist functions  $\mu_{ij}$  as defined in (13), and that the conditions (I)-(II) of Theorem 2 are satisfied. Then, the origin of the closed-loop system (20) under the effect of control input (24) and the observer (21) is an FTS equilibrium.*

*Proof.* We first show that there exists  $T_1 < \infty$  such that for all  $t \geq T_1$ ,  $\hat{x}(t) = x(t)$ . Note that the origin is the only equilibrium of (23). From the analysis in Theorem 2, we know that the conditions (I) and (II) of Theorem 1 are satisfied. The observation-error dynamics for mode  $\sigma_0$  reads:

$$\dot{e} = \begin{bmatrix} e_2 - l_1 \text{sign}(e_1)|e_1|^{\alpha_1} \\ e_3 - l_2 \text{sign}(e_1)|e_1|^{\alpha_2} \\ \vdots \\ e_n - l_{n-1} \text{sign}(e_1)|e_1|^{\alpha_{n-1}} \\ -l_n \text{sign}(e_1)|e_1|^{\alpha_n} \end{bmatrix}. \quad (25)$$

Now, using [20, Theorem 10], we obtain that the origin is an FTS equilibrium for (25), i.e., for mode  $\sigma_0$  of (23). From [20, Lemma 8], we also know that (25) is homogeneous with degree of homogeneity  $d = \alpha - 1 < 0$ . Hence, using [21, Theorem 7.2], we obtain that there exists a Lyapunov function  $V_o$  satisfying  $\dot{V}_o \leq -cV_o^\beta$  where  $c > 0$  and  $0 < \beta < 1$ . Hence, condition (iii) of Theorem 1 is also satisfied. From the proof of Theorem 2, we obtain that the condition (iv) of Theorem 1 is also satisfied. Hence, we obtain that the origin of (23) is an FTS equilibrium. Thus, there exists  $T_1 < \infty$  such that for all  $t \geq T_1$ ,  $\hat{x}(t) = x(t)$ . Therefore, for  $t \geq T_1$ , the control input satisfies  $u = u(\hat{x}) = u(x)$ . It is easy to verify that the origin is the only equilibrium for (20) under the effect of control input (24) with  $x = \hat{x}$ . From [21, Proposition 8.1], we know that the origin of the closed-loop trajectories for mode  $\sigma = \sigma_0$  is FTS. Hence, repeating the same set of arguments as above, we obtain that there exists  $T_2 < \infty$  such that for all  $t \geq T_1 + T_2$ , the closed-loop trajectories of (20) satisfy  $x(t) = 0$ .  $\square$

#### IV. SIMULATIONS

We present a numerical example to demonstrate the efficacy of the proposed method. Note that the simulation results have been obtained by discretizing the continuous-time dynamics using Euler discretization. We use a step size of  $dt = 10^{-3}$ , and run the simulations till the norm of the states drops below  $10^{-15}$ . The example considers a switched linear control system of the form (20) with four modes such that only one mode is both controllable and observable. We design an FTS output controller for the considered switched system, and demonstrate that the closed-loop trajectories reach the origin despite the presence of unobservable and uncontrollable modes. We model the system as a double integrator:

$$\dot{z} = \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = A_\sigma \begin{bmatrix} x \\ v \end{bmatrix} + B_\sigma u, \quad y = C_\sigma \begin{bmatrix} x \\ v \end{bmatrix},$$

where  $x, v \in \mathbb{R}^2$  are the position and the velocity of the system, respectively, and  $u \in \mathbb{R}^2$  the acceleration controller, with  $\sigma \in \{1, 2, 3, 4\}$ , and assume that mode  $\sigma_0 = 4$  is both controllable and observable, while the rest of the modes are either uncontrollable and unobservable. The simulation parameters are:

- $N = 4$ ,  $\alpha = 0.9$ ,  $\beta = .9$   $k_1 = 20$  and  $k_2 = 10$ ;
- $A = \begin{bmatrix} 0_2 & I_2 \\ 0_2 & 0_2 \end{bmatrix}$ ,  $B_1 = B_3 = \begin{bmatrix} 0_2 \\ 0_2 \end{bmatrix}$ ,  $B_2 = B_4 = \begin{bmatrix} 0_2 \\ I_2 \end{bmatrix}$  and  $C_1 = C_2 = [0_2 \ 0_2]$ ,  $C_3 = C_4 = [I_2 \ 0_2]$  with  $I_2 \in \mathbb{R}^{2 \times 2}$  being the identity matrix and  $0_2 \in \mathbb{R}^{2 \times 2}$  being a matrix consisting of zeros.
- $V_i(z) = z^T z$  for  $i = 1, 2, 3$ , and  $V_4(z) = \frac{k_2}{2\alpha} \|x\|^{2\alpha} + \frac{1}{2} \|v\|^2$  and

$$\mu_{ij}(z) = \begin{cases} -\|z\|^2, & i \in \{1, 2, 4\}; \\ 0, & i \in \{3, 5\}; \end{cases}, \quad \forall j \in \sigma.$$

Note that mode 1 is both unobservable and uncontrollable, mode 2 is unobservable and controllable, mode 3 is observable and uncontrollable and mode 4 is both controllable and observable. The generalized Lyapunov candidates  $V_i$  are quadratic and hence satisfy condition (i) of Theorem 1. Mode 4 being stable satisfies condition (ii) with  $\alpha_2 = 0$ . The modes 1, 2 and 3 are active only for a finite time, and therefore satisfy condition (ii) with  $\alpha_2 = k\|z_0\|^2$  for some  $k > 0$ .

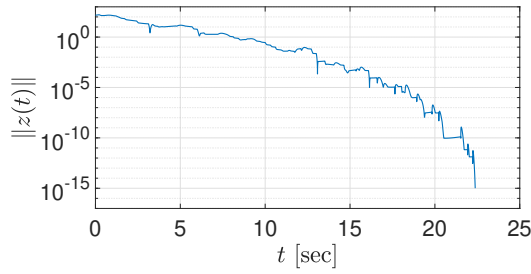


Fig. 2. The norm of the system states  $\|z(t)\|$  for the closed-loop system for  $z(0) = [75.8 \ -72.5 \ 73.4 \ -72.4]^T$ .

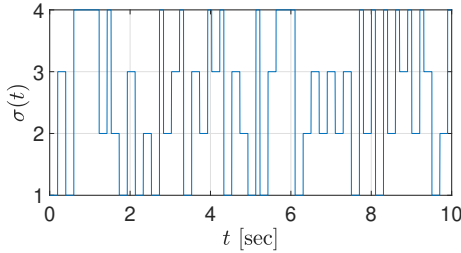


Fig. 3. Switching signal for the closed-loop system.

Condition (iv) is satisfied by designing the switching signal as discussed in Section III-B. Figure 2 plots the norm of the states  $\|z(t)\|$  of the closed-loop system; it can be seen that the system trajectory reaches the origin within a finite amount of time. The figure demonstrates that satisfaction of the conditions of Theorem 1 leads to FTS despite the presence of unstable modes. Figure 3 plots the switching signal, illustrating that the unobservable and uncontrollable modes are active for a significant amount of time (about 73% of the total simulation time). Note that the activation of the unstable modes is meaningful (opposed to not activating them at all) since operation under them might be relevant to, and dictated by, the practical problem, as highlighted in the example in Section I.

We simulated the system (23) for 500 different initial conditions  $e(0)$ . Figure 4 shows the actual cumulative time  $T_F$  that the mode  $F$  is active for various initial errors  $\|e(0)\|$ , before the observation error converges to zero. It can be seen that  $T_F(\|e(0)\|) \leq \gamma(\|e(0)\|)$  for all  $e(0)$ , where  $\gamma(\|e(0)\|) = 10\|e(0)\|^{2-2\alpha} \in \mathcal{K}$ . This shows that in practice, the FTS mode can be active for a smaller amount of time than the estimate in Theorem 1.

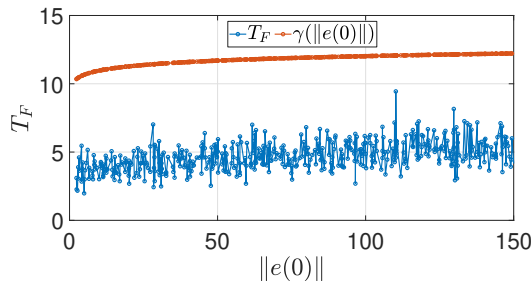


Fig. 4. Cumulative active time for mode  $F$ . The blue line shows the actual time  $T_F$  the mode  $F$  is active while red line is the required active time  $\gamma(\|e(0)\|)$ .

## V. CONCLUSIONS

In this paper, we studied the FTS of a class of switched systems. We showed that under some mild conditions on the bounds on the difference of the values of Lyapunov functions, if the FTS mode is active for a sufficient cumulative time, then the origin of the switched system is FTS. As an application of the theoretical results, we designed an FTS output feedback for a class of linear switched systems where only one of the modes is both controllable and observable. As future research, we plan to investigate how the results presented in this paper can be used for the systematic control synthesis of switched systems under spatiotemporal constraints, requiring closed-loop trajectories to reach a given goal set within a finite time.

## REFERENCES

- [1] D. Liberzon, *Switching in Systems and Control*. Springer Science & Business Media, 2003.
- [2] H. Ishii and B. Francis, "Stabilizing a linear system by switching control with dwell time," *IEEE Transactions on Automatic Control*, vol. 47, no. 12, pp. 1962–1973, 2002.
- [3] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, "Stability criteria for switched and hybrid systems," *SIAM review*, vol. 49, no. 4, pp. 545–592, 2007.
- [4] M. S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 475–482, 1998.
- [5] J. Zhao and D. J. Hill, "On stability, L2-gain and H-infinity control for switched systems," *Automatica*, vol. 44, no. 5, pp. 1220–1232, 2008.
- [6] X. Zhao, L. Zhang, P. Shi, and M. Liu, "Stability of switched positive linear systems with average dwell time switching," *Automatica*, vol. 48, no. 6, pp. 1132–1137, 2012.
- [7] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal on Control and Optimization*, vol. 38, no. 3, pp. 751–766, 2000.
- [8] X. Liu, D. W. Ho, Q. Song, and J. Cao, "Finite-/fixed-time robust stabilization of switched discontinuous systems with disturbances," *Nonlinear Dynamics*, vol. 90, no. 3, pp. 2057–2068, 2017.
- [9] Y. Li and R. G. Sanfelice, "A robust finite-time convergent hybrid observer for linear systems," in *52nd IEEE Conference on Decision and Control*, Dec 2013, pp. 3349–3354.
- [10] H. Rios, J. Davila, and L. Fridman, "State estimation on switching systems via high-order sliding modes," in *Hybrid Dynamical Systems*. Springer, 2015, pp. 151–178.
- [11] Y. Orlov, "Finite time stability and robust control synthesis of uncertain switched systems," *SIAM Journal on Control and Optimization*, vol. 43, no. 4, pp. 1253–1271, 2004.
- [12] B. Zhang, "On finite-time stability of switched systems with hybrid homogeneous degrees," *Mathematical Problems in Engineering*, vol. 2018, 2018.
- [13] J. Fu, R. Ma, and T. Chai, "Global finite-time stabilization of a class of switched nonlinear systems with the powers of positive odd rational numbers," *Automatica*, vol. 54, pp. 360–373, 2015.
- [14] F. Bejarano, A. Pisano, and E. Usai, "Finite-time converging jump observer for switched linear systems with unknown inputs," *Nonlinear Analysis: Hybrid Systems*, vol. 5, no. 2, pp. 174–188, 2011.
- [15] Y. Li and R. G. Sanfelice, "Finite time stability of sets for hybrid dynamical systems," *Automatica*, vol. 100, pp. 200–211, 2019.
- [16] Z. Zuo and L. Tie, "Distributed robust finite-time nonlinear consensus protocols for multi-agent systems," *International Journal of Systems Science*, vol. 47, no. 6, pp. 1366–1375, 2016.
- [17] Y.-E. Wang, H. R. Karimi, and D. Wu, "Conditions for the stability of switched systems containing unstable subsystems," *IEEE Transactions on Circuits and Systems II: Express Briefs*, 2018.
- [18] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "Introducing SOSTOOLS: a general purpose sum of squares programming solver," in *41st IEEE Conference on Decision and Control*, Dec 2002, pp. 741–746.
- [19] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Systems Magazine*, vol. 19, no. 5, pp. 59–70, 1999.
- [20] W. Perruquetti, T. Floquet, and E. Moulay, "Finite-time observers: application to secure communication," *IEEE Transactions on Automatic Control*, vol. 53, no. 1, pp. 356–360, 2008.
- [21] S. P. Bhat and D. S. Bernstein, "Geometric homogeneity with applications to finite-time stability," *Mathematics of Control, Signals, and Systems*, vol. 17, no. 2, pp. 101–127, 2005.