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Finite-Time Resilient Formation Control with Bounded Inputs

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Abstract— **In this paper we consider the problem of a multi-agent system achieving a formation in the presence of misbehaving or adversarial agents. We introduce a novel continuous time resilient controller to guarantee that normally behaving agents can converge to a formation with respect to a set of leaders. The controller employs a norm-based filtering mechanism, and unlike most prior algorithms, also incorporates input bounds. In addition, the controller is shown to guarantee convergence in finite time. A sufficient condition for the controller to guarantee convergence is shown to be a graph theoretical structure which we denote as Resilient Directed Acyclic Graph (RDAG). Further, we employ our filtering mechanism on a discrete time system which is shown to have exponential convergence. Our results are demonstrated through simulations.**

I. Introduction

The study of resilient control in the presence of adversarial agents is a rapidly growing field. An ever-growing amount of cyber attacks has led to increasing attention on algorithms that guarantee safety and security despite the influence of faults and malicious behavior. Controllers that protect against adversarial actions are especially critical in distributed systems where agents may have limited power, computational capabilities, and knowledge of the system as a whole.

The problem of agents achieving formation with respect to a leader or set of leaders has been well-studied in the literature under the assumption that all agents are behaving (see [1] and its references). However, it is well known that the introduction of one faulty or misbehaving agent can disrupt the performance of the entire network. The literature has addressed this problem when agents have simply crashed, have actuator or sensor faults, or have malicious intent [2]–[4]. Much work remains to be done in this area however, especially when the misbehaving agents have malicious intent rather than simply being subject to faults.

A certain group of resilient consensus algorithms based on a filtered-mean or median based approach have gained traction recently in the literature. These algorithms include the W-MSR [5], ARC-P [6], SW-MSR [7], DP-MSR [8], LFRE [9], and MCA [10] algorithms, which have all been used for resilient consensus. There are a few limitations to these prior results. One limitation is that no upper bound is assumed on agents' maximum inputs. Many systems with agents coming to consensus on physical states are modeled as having such a bound. In addition, the ARC-P algorithm has been shown to have asymptotic convergence, but to the best of authors' knowledge a precise convergence rate has not been proven.

Finite-time consensus has been a popular field of research recently [11]–[15] and little prior work has addressed this topic in the case of resilient controllers. The consensus algorithm in [16] does consider a resilient algorithm with finite time convergence and bounded inputs. However, it considers undirected graphs where all misbehaving agents must be connected only to agents which are guaranteed to be cooperative. The analysis in this paper considers directed graphs and assumes a different adversary model, where all agents are vulnerable to attacks. As we will show, our method does not require the knowledge of a set of agents invulnerable to misbehavior.

Our contributions are as follows: (i) We introduce a novel continuous finite-time controller that allows agents to achieve formations in the presence of adversarial agents. The controller employs a novel filtering mechanism based on the norm of the difference between agents' states. (ii) We prove that this controller guarantees convergence with bounded inputs. (iii) We define novel conditions for the filtering timing and input weights which ensure that agents can remain in formation even with a dwell time in the filtering mechanism. (iv) We show that the norm-based filtering and bounded input elements of our continuous-time controller can be used in a similar resilient discrete-time system, which is proven to have exponential convergence.

Our paper is outlined as follows: in Section II we outline our notation and give the problem formulation; in Section III and IV we present our main results on resiliently achieving formation in continuous and discrete time, respectively; in Section V we present simulations demonstrating our results; and our conclusions and thoughts on future work are summarized in Section VI.

II. Modeling and Problem Formulation

A. Notation

We denote a directed graph (digraph) as $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \{1, ..., n\}$ denoting the vertex set, or agent set, of the graph and $\mathcal{E}$ denoting the edge set of the graph. A directed edge is denoted as $(j, i) \in \mathcal{E} : i, j \in \mathcal{V}$, which implies that i is able to sense or receive information from agent j. In this case we say that agent j is an in-neighbor of i and i is an out-neighbor of j. The set of in-neighbors of agent i is denoted $\mathcal{V}_i = \{j : (j, i) \in \mathcal{E})\}$. Three subsets of $\mathcal{V}$ are considered in this paper: leader agents $\mathcal{L}$, adversarial agents

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$\mathcal{A}$, and normal agents $\mathcal{N}$. These subsets will be described in more detail in section II-B. We denote $\mathcal{A}_i = \mathcal{V}_i \bigcap \mathcal{A}$, i.e. the set of adversarial agents in the in-neighbour set of agent i. The in-neighbors which are *not* filtered out are denoted $\mathcal{R}_i \subseteq \mathcal{V}_i$. For brevity of notation, we will denote $\mathcal{R}_i^{\mathcal{N}} = \mathcal{R}_i \setminus (\mathcal{A}_i \bigcap \mathcal{R}_i)$ and $\mathcal{R}_i^{\mathcal{A}} = \mathcal{A}_i \bigcap \mathcal{R}_i$, which implies $\mathcal{R}_i = \mathcal{R}_i^{\mathcal{N}} \bigcup \mathcal{R}_i^{\mathcal{A}}$. We also denote $R_i \triangleq |\mathcal{R}_i(t)|$. The cardinality of a set S is written as $|S|$, the set of integers as $\mathbb{Z}$, the set of integers greater than or equal to zero as $\mathbb{Z}_{\geq 0}$, the natural numbers as $\mathbb{N}$ and the set of non-negative reals as $\mathbb{R}_+$. Finally, $\|\cdot\|$ denotes any p-norm defined on $\mathbb{R}^n$. The control protocols in this paper involve a process in which each agent $i \in \mathcal{N}$ filters out a subset of its in-neighbors $\mathcal{V}_i$. The details are given in sections III-A and IV-A.

B. Problem Definition

Consider a time-invariant digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ of n agents with states $\boldsymbol{p}_i \in \mathbb{R}^n$. Each agent $i \in \mathcal{V}$ has the system model

$$\delta \boldsymbol{p}_i(t) = \boldsymbol{u}_i(t), \tag{1}$$

where $\delta \boldsymbol{p}_i(t)$ denotes the time derivative $\dot{\boldsymbol{p}}_i$ for the case of continuous-time system and the time-difference $\boldsymbol{p}_i[t+1] - \boldsymbol{p}_i[t]$ for the case of discrete-time systems. The variable $\boldsymbol{u}_i$ is the input to agent i, which will be explained in sections III and IV respectively for continuous- and discrete-time system.

There is much prior literature on formation control problems involving a set of leaders to which the rest of the network converges. We assume that a subset of the agents $\mathcal{L} \subset \mathcal{V}$ are designated to behave as leaders. However, these leaders are not invulnerable to attacks, implying $(\mathcal{L} \bigcap \mathcal{A})$ may possibly be nonempty. Any nodes which are neither leaders nor adversarial are designated as normal nodes $\mathcal{N} \subset \mathcal{V}$. In all, $\mathcal{N} \bigcup \mathcal{L} \bigcup \mathcal{A} = \mathcal{V}$.

We assume that prescribed constant formation vectors $\boldsymbol{\xi}_i \in \mathbb{R}^n$ have been specified for these agents. Each $\boldsymbol{\xi}_i \in \mathbb{R}^n$ represents agent i's desired formational offset from a group reference point. The formation offsets of the entire network is written as $\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\xi}_1^T & \dots & \boldsymbol{\xi}_n^T \end{bmatrix}^T$. As outlined in [17, Chapter 6], we define the variable $\boldsymbol{\tau}_i(t) = \boldsymbol{p}_i(t) - \boldsymbol{\xi}_i$. If non-adversarial agents come to formation on their values of $\boldsymbol{\tau}_i(t)$, i.e. $\|\boldsymbol{\tau}_i(t) - \boldsymbol{\tau}_j(t)\| \to 0 \; \forall i, j \in (\mathcal{L} \bigcup \mathcal{N}) \backslash \mathcal{A}$ then they have achieved formation. In essence, the agents will have achieved consensus on the center of formation. The behaving leaders are assumed to be maintaining their $\boldsymbol{\tau}$ values at some arbitrary point $\boldsymbol{\tau}_L$. The goal of this paper is to design a resilient control protocol such that all the normal behaving agents can come to formation at $\boldsymbol{\tau}_L$. We assume that each agent i is able to obtain the time-varying relative vectors $\boldsymbol{\tau}_j(t) - \boldsymbol{\tau}_i(t)$ for all $j \in \mathcal{V}_i$. Two ways in which this might be accomplished include each agent i measuring this vector via on-board sensors or calculating it by receiving transmitted messages from each $j \in \mathcal{V}_i$. In the former case, it is required that agents share a common reference orientation, and in the latter both a reference orientation and a common origin point.

We assume that there exists a subset of the agents $\mathcal{A} \subset \mathcal{V}$ that is adversarial, and that $\mathcal{A}$ is an F-*total* set; i.e. for any $i \in (\mathcal{V} \backslash \mathcal{A}),\; |\mathcal{V}_i \bigcap \mathcal{A}| \leq F$ ([5]). Any adversarial agent $k \in \mathcal{A}$ may attempt to prevent its normal out-neighbors from coming to formation by manipulating the value of $\boldsymbol{\tau}_k(t)$ received by its out-neighbors. Two ways in which this may occur include physical or communication misbehavior. Explicitly, an agent misbehaves if at any time $t \geq t_0$ it applies a different control law than the nominal one in the former case, or by sending false information in the latter. In either case, this misbehavior is modeled as normal agent i obtaining the time-varying relative vector $\boldsymbol{\tau}_k(t) - \boldsymbol{\tau}_i(t)$ where the adversarial dynamics of the value of $\boldsymbol{\tau}_k(t)$ received by any normal agent i are

$$\delta \boldsymbol{\tau}_k(t) = \boldsymbol{f}_{k,m}(t), \tag{2}$$

where $\boldsymbol{f}_{k,m}(t)$ is the adversarial input. This adversarial behavior is malicious ([5]) in the sense that each out-neighbor of k receives the same misbehavior. As outlined in [6], since in the continuous time case each normal agent will have continuous state trajectories, any discontinuity in an adversarial agent's transmitted signal could expose its misbehavior to the network. Hence we assume that in the continuous time case, the time-varying relative vector $\boldsymbol{\tau}_k(t) - \boldsymbol{\tau}_i(t)$ obtained by any normal agent $i \in \mathcal{N}$ from any adversary $k \in \mathcal{A}$ is continuous. The assumption of continuity of $\boldsymbol{\tau}_k(t)$ is also made for the case where agents make on-board measurements.

C. Graph Theoretical Conditions

Our method employs a graph-theoretical structure which we call a Resilient Directed Acyclic Graph (RDAG). This structure is a special case of a class of graphs called Mode Estimated Directed Acyclic Graphs (MEDAGs) [18], and is defined as follows:

Definition 1: A digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ is a *Resilient Directed Acyclic Graph* (RDAG) with parameter r if it satisfies the following properties for an integer $r \geq 0$:

1) There exists a partitioning of $\mathcal{V}$ into sets $\mathcal{S}_0, \dots, \mathcal{S}_m \subset \mathcal{V},\; m \in \mathbb{Z}$ such that $|\mathcal{S}_j| \geq r$ for all $0 \leq j \leq m$.
2) For each $i \in \mathcal{S}_j,\; 1 \leq j \leq m,\; \mathcal{V}_i \subseteq \bigcup_{k=0}^{j-1} S_k$

Intuitively, an RDAG is a graph defined by successive subsets of agents $\mathcal{S}_j$. Agents in each subset only have in-neighbors from preceding subsets. The purpose of an RDAG is to introduce enough edge redundancy to ensure the existence of an unfiltered directed path of behaving nodes from the leaders to each normal agent. This can be achieved by designating all agents in the set $\mathcal{S}_0$ to behave as leaders, i.e. $\mathcal{S}_0 = \mathcal{L}$. In our analysis, we consider RDAGs with parameter $r \geq 3F + 1$. As we will show, an RDAG of this form is a sufficient condition implying that normal agents applying our filtering methods and controllers will converge to the leaders. A method exists by which RDAGs can be constructed from existing graph topologies even in the presence of adversaries ([9]). This method involves agents successively receiving in a resilient manner and rebroadcasting a communication signal initiated by the set of leaders, identifying their own set and the agents in the preceding set, and then restricting their in-neighbor set to only agents in the preceding set. In particular, an RDAG can be constructed from an initial graph that is strongly robust with respect to a subset $\mathcal{S} \subset \mathcal{V}$. An example of such a graph is a k-circulant graph [19]. The existence of an RDAG graph structure does not guarantee that normal agents are able to identify adversarial agents. Rather, the edge redundancy guarantees that each normal agent has enough behaving in-neighbors to still achieve formation under the proposed controllers.

III. Continuous-time System

A. Filtering Algorithm and Control Law

In the continuous time setting, each agent applies Algorithm 1 at every time instance $t = m\epsilon_d$, where $\epsilon_d > 0$ is defined later in this section.

Algorithm 1 Continuous-Time Filtering

procedure UpdateFilteredList
  Calculate $\tau_{ij} = \|\boldsymbol{\tau}_j - \boldsymbol{\tau}_i\| \;\; \forall j \in \mathcal{V}_i$
  if $t = m\epsilon_d, \;\; m \in \mathbb{Z}_{\geq 0}, \;\; \epsilon_d > 0$ **then**
    Sort τ_{ij} values such that $\tau_{ij_1} \geq \ldots \geq \tau_{ij_{|\mathcal{V}_i|}}$
    $\mathcal{R}_i(t) \leftarrow \{j : \tau_{ij} \in \{\tau_{ij_{F+1}}, \ldots, \tau_{ij_{|\mathcal{V}_i|}}\}\}$

The dynamics of continuous time $\boldsymbol{\tau}(t)$ are given as:

$$\dot{\boldsymbol{\tau}}_i(t) = \dot{\boldsymbol{p}}_i(t) - \dot{\boldsymbol{\xi}}_i = \boldsymbol{u}_i(t). \tag{3}$$

We will sometimes omit the argument t for the sake of brevity when the dependence on t is clear from the context. We assume that the speed of each agent i is bounded above by u_M, i.e. $\|\boldsymbol{u}_i(t)\| \leq u_M$ for all $t \geq 0$. Under this constraint, the saturation function is defined as

$$\sigma_i(t) = \min\{\|\boldsymbol{u}_i^p(t)\|, u_M\}, \tag{4}$$

$$\boldsymbol{u}_i^p(t) = \sum_{j \in \mathcal{R}_i(t)} w_{ij}(t)(\boldsymbol{\tau}_j(t) - \boldsymbol{\tau}_i(t))\|\boldsymbol{\tau}_j - \boldsymbol{\tau}_i\|^{\alpha-1}, \tag{5}$$

where $0 < \alpha < 1$. To simplify the notation, define the term $\gamma_i(t) = \frac{\sigma_i(t)}{\|\boldsymbol{u}_i^p\|}$. With this saturation function[1], inspired from the control law used in [20] and using results from [21], we define the continuous time control law as:

$$\boldsymbol{u}_i(t) = \sum_{j \in \mathcal{R}_i(t)} \gamma_i(t) w_{ij}(t)(\boldsymbol{\tau}_j - \boldsymbol{\tau}_i)\|\boldsymbol{\tau}_j - \boldsymbol{\tau}_i\|^{\alpha-1} \tag{6}$$

where $0 < \alpha < 1$. It can be verified from (6) that $\|\boldsymbol{u}_i(t)\| \leq u_M$ for all $t \geq 0$ and that the control input goes to zero as agent i goes to its equilibrium.[2] Note that for $\alpha = 1$, the control law (6) is same as the traditional formation control law (see [22] for example), while for $\alpha = 0$, we obtain a control law similar to the one introduced in [23]. We make use of this type of controller to not only ensure that $\boldsymbol{\tau}_i$ converges to $\boldsymbol{\tau}_L$, but does so in finite time.

As opposed to [24], this protocol is designed such that agents do not update their filtered list $\mathcal{R}_i(t)$ at every time instance t, but instead only at time instances $t_1, t_2, t_3, ...$ while keeping it constant during the interval (t_l, t_{l+1}). Each of these intervals have constant length, i.e. $t_{l+1} - t_l = \epsilon_d$ for all $l \in \{1, 2, 3, ...\}$ where $\epsilon_d > 0$ is a user-defined small, positive constant. The weights $w_{ij}(t)$ for all $i \in \mathcal{N}$ are designed such that malicious agents are not able to exploit this behavior of $\mathcal{R}_i(t)$. Let $\Omega_i(t)$ be the set of in-neighbour agents whose $\boldsymbol{\tau}$ vectors are NOT equal to that of agent i, i.e. $\Omega_i(t) = \{j \in \mathcal{V}_i : \|\boldsymbol{\tau}_j - \boldsymbol{\tau}_i\| > 0\}$. Then for all $i \in \mathcal{N}$, we define the control weights $w_{ij}(t)$ for all $j \in \mathcal{R}_i(t)$ as

$$w_{ij}(t) = \begin{cases} 0, & |\Omega_i(t)| \leq F, \\ \frac{1}{R_i}, & |\Omega_i(t)| > F. \end{cases} \tag{7}$$

To the authors' best knowledge, this choice of control weights have never been introduced in the prior literature. Intuitively, this implies that each normal agent i will have a velocity of zero if its $\boldsymbol{\tau}$ is co-located with the $\boldsymbol{\tau}$ of all but at most F of its in-neighbors. We impose this constraint to ensure that when all normal agents' $\boldsymbol{\tau}$ values have converged to $\boldsymbol{\tau}_L$, the malicious agents cannot perturb them away from $\boldsymbol{\tau}_L$ during the dwell time. This could happen, for example, if for some $i \in \mathcal{N}$, $\|\tau_i - \tau_k\| = 0$ at all $t = m\epsilon_d$ and $\|\tau_i - \tau_k\| > 0$ for time $t \in (m\epsilon_d, (m+1)\epsilon_d)$, where $k \in \mathcal{A}_i$, $m \in \mathbb{Z}_{\geq 0}$. Since $\mathcal{R}_i(t)$ is constant for each $t \in [m\epsilon_d, (m+1)\epsilon_d)$, the malicious agents would not be filtered out by agent i. The properties we impose on the weights prevent the malicious agents from steering the normal agents away during such period.

Theorem 1: For each agent $i \in \mathcal{N}$, $|\Omega_i(t)| \leq F$ (or, $w_{ij}(t) = 0 \; \forall \; j \in \mathcal{R}_i$) for all $t \geq t_i$, if and only if $\boldsymbol{\tau}_i(t) = \boldsymbol{\tau}_L$ for all $t \geq t_i$, for some time t_i.

Proof: Sufficiency: Assume that there exists some time instant t_i such that for all future times $t \geq t_i$, $\|\boldsymbol{\tau}_i(t) - \boldsymbol{\tau}_L\| \equiv 0$. This can only happen if all the filtered in-neighbors of the agent i (i.e. $j \in \mathcal{R}_i$) are at $\boldsymbol{\tau}_L$. To see why this is true, assume that there exists a filtered in-neighbour of agent i which is not at $\boldsymbol{\tau}_L$. Then, by the virtue of the control law (6), agent i would have a non-zero control input $\boldsymbol{u}_i(t)$, which is a contradiction to the assumption that agent stays at the point $\boldsymbol{\tau}_L$. Hence, all its filtered in-neighbours are at the point $\boldsymbol{\tau}_L$. Since we assume that there are at most F agents in the filtered set $\mathcal{V}_i \backslash \mathcal{R}_i$, we have that at most these F agents may not be at τ_L, i.e. $|\Omega_i(t)| \leq F$ and $w_{ij}(t) = 0 \, \forall j \in \mathcal{R}_i$.

Necessity: We prove this by contradiction. Let us assume that there exist $\boldsymbol{\tau}^* \neq \boldsymbol{\tau}_L$ and a time t_i such that $\boldsymbol{\tau}_i(t) = \boldsymbol{\tau}^*$ and in addition we have that $|\Omega_i(t)| \leq F$ for all $t \geq t_i$. Let us assume that $i \in \mathcal{S}_p$. Since $|\mathcal{V}_i| \geq 3F+1$ and $|\Omega_i(t)| \leq F$, there are at least $2F+1$ in-neighbors which are also staying at $\boldsymbol{\tau}^*$. This implies that there is at least one normal behaving agent in the in-neighbour set of agent i in the set $\bigcup_{l=0}^{p-1} \mathcal{S}_l$, which stays at $\boldsymbol{\tau}^*$. This in turn means that one of its normal behaving in-neighbors in the set $\bigcup_{l=0}^{p-2} \mathcal{S}_l$ stays identically at $\boldsymbol{\tau}^*$. Using this argument recursively, we have that there exists a normal in-neighbor in the set $\mathcal{S}_0$, which stays identically at the location $\boldsymbol{\tau}^*$. Since all the normal behaving in-neighbors $\mathcal{S}_0$ stay at $\boldsymbol{\tau}_L$, this contradicts the assumption $\boldsymbol{\tau}^* \neq \boldsymbol{\tau}_L$. Hence, we obtain $\boldsymbol{\tau}_i^* = \boldsymbol{\tau}_L$, and that $|\Omega(t)| \leq F$ for all $t \geq t_i$ only if $\boldsymbol{\tau}_i(t) = \boldsymbol{\tau}_L$ for all $t \geq t_i$. ■

[1] For all $t \geq 0$, $0 \leq \gamma_i(t) \leq 1$. Note that if the distances of agent from its in-neighbours $j \in \mathcal{R}_i$ are finite, then $\gamma_i(t)$ is strictly positive.

[2] As $\boldsymbol{\tau}_j \to \boldsymbol{\tau}_i$, term $(\boldsymbol{\tau}_j - \boldsymbol{\tau}_i)\|\boldsymbol{\tau}_j - \boldsymbol{\tau}_i\|^{\alpha-1} \to 0$ for $\alpha > 0$

B. Convergence Analysis

We now prove that under the control law (6), filtering Algorithm 1, and the definition of control weights w_{ij} in (7), all the normal behaving agents achieve formation in finite time, despite the presence of adversarial agents. First, we show that for each normal agent $i \in \mathcal{S}_1$, $\|\boldsymbol{\tau}_i(t) - \boldsymbol{\tau}_L\|$ converges to zero in finite time:

Lemma 1: Consider a digraph $\mathcal{D}$ which is an RDAG with parameter $3F+1$, where $\mathcal{S}_0 = \mathcal{L}$ and $\mathcal{A}$ is an F-total set. For each normal agent $i \in \mathcal{S}_1$, $\boldsymbol{\tau}_L$ is a globally finite-time stable equilibrium for the closed-loop dynamics (3)-(7).

Proof: Choose the candidate Lyapunov function $V(\boldsymbol{\tau}_i) = \frac{1}{2}\|\boldsymbol{\tau}_i - \boldsymbol{\tau}_L\|^2$. Since $\dot{\boldsymbol{\tau}}_i$ is piece-wise continuous in each interval (t_l, t_{l+1}), the trajectory $\boldsymbol{\tau}_i(t)$ is piecewise differentiable in each such interval. Let $\dot{\boldsymbol{\tau}}_i(t_{l+1}^-)$ and $\dot{\boldsymbol{\tau}}_i(t_{l+1}^+)$ denote the value of the vector $\dot{\boldsymbol{\tau}}_i$ just before and after the filtering at time instant t_{l+1}, respectively. Because the right hand side of (6) is bounded at the beginning of each interval, the upper right Dini derivative is defined for $\boldsymbol{\tau}_i(t)$ everywhere, and takes values as

$$D^+(V(\boldsymbol{\tau}_i))(t) = \begin{cases} \nabla V(\boldsymbol{\tau}_i)\dot{\boldsymbol{\tau}}_i(t), & t_l \leq t < t_{l+1}, \\ \nabla V(\boldsymbol{\tau}_i)\dot{\boldsymbol{\tau}}_i(t_{l+1}^+), & t = t_{l+1}. \end{cases},$$

For the worst case, assume that there are F adversarial agents and $R_i - F$ leaders in the filtered list $\mathcal{R}_i$. This requires that the adversarial agent should satisfy $\|\boldsymbol{\tau}_i - \boldsymbol{\tau}_j\| \leq \|\boldsymbol{\tau}_i - \boldsymbol{\tau}_L\|$ for all $j \in \mathcal{A}_i$ and for all $t \geq 0$, otherwise agent j would be filtered out as per Section III-A. Using this and taking the upper right Dini-derivative of the candidate Lyapunov function along the closed loop trajectories of (3), we have:

$$\begin{aligned} D^+(V(\boldsymbol{\tau}_i)) &= (\boldsymbol{\tau}_i - \boldsymbol{\tau}_L)^T \sum_{j \in \mathcal{R}_i^{\mathcal{N}}} \gamma_i w_{ij}(\boldsymbol{\tau}_j - \boldsymbol{\tau}_i)\|\boldsymbol{\tau}_j - \boldsymbol{\tau}_i\|^{\alpha-1} \\ &+ (\boldsymbol{\tau}_i - \boldsymbol{\tau}_L)^T \sum_{j \in \mathcal{R}_i^{\mathcal{A}}} \gamma_i w_{ij}(\boldsymbol{\tau}_j - \boldsymbol{\tau}_i)\|\boldsymbol{\tau}_j - \boldsymbol{\tau}_i\|^{\alpha-1} \\ &= \gamma_i \frac{R_i - F}{R_i}(\boldsymbol{\tau}_i - \boldsymbol{\tau}_L)^T(\boldsymbol{\tau}_L - \boldsymbol{\tau}_i)\|\boldsymbol{\tau}_L - \boldsymbol{\tau}_i\|^{\alpha-1} \\ &+ (\boldsymbol{\tau}_i - \boldsymbol{\tau}_L)^T \sum_{j \in \mathcal{R}_i^{\mathcal{A}}} \gamma_i w_{ij}(\boldsymbol{\tau}_j - \boldsymbol{\tau}_i)\|\boldsymbol{\tau}_j - \boldsymbol{\tau}_i\|^{\alpha-1} \end{aligned}$$

Since $\|\boldsymbol{\tau}_i - \boldsymbol{\tau}_j\| \leq \|\boldsymbol{\tau}_i - \boldsymbol{\tau}_L\|$ for all $j \in \mathcal{R}_i^{\mathcal{A}}$, we have:

$$\begin{aligned} D^+(V(\boldsymbol{\tau}_i)) &\leq -\gamma_i \frac{R_i - F}{R_i}\|\boldsymbol{\tau}_i - \boldsymbol{\tau}_L\|^{1+\alpha} \\ &+\gamma_i \sum_{j \in \mathcal{R}_i^{\mathcal{A}}} w_{ij}\|\boldsymbol{\tau}_i - \boldsymbol{\tau}_L\|\|\boldsymbol{\tau}_j - \boldsymbol{\tau}_i\|\|\boldsymbol{\tau}_j - \boldsymbol{\tau}_i\|^{\alpha-1} \\ &\leq -\gamma_i \frac{R_i - F}{R_i}\|\boldsymbol{\tau}_i - \boldsymbol{\tau}_L\|^{1+\alpha} \\ &+\gamma_i \frac{F}{R_i - F}\|\boldsymbol{\tau}_i - \boldsymbol{\tau}_L\|\|\boldsymbol{\tau}_L - \boldsymbol{\tau}_i\|\|\boldsymbol{\tau}_L - \boldsymbol{\tau}_i\|^{\alpha-1} \end{aligned}$$
$$\Rightarrow D^+(V(\boldsymbol{\tau}_i)) \leq -cV(\boldsymbol{\tau}_i)^\beta,$$

where $\beta = \frac{1+\alpha}{2} < 1$. Note that $D^+(V(\boldsymbol{\tau}_i)) \leq 0$ which means that the Lyapunov candidate $V(\boldsymbol{\tau}_i(t))$ is bounded by $V(\boldsymbol{\tau}_i(0))$. This implies that the agent i remains at a bounded distance from the leaders. Also, if any adversarial agent's state moves further away, by the filtering algorithm, they would be filtered out. Hence, each term in $\boldsymbol{u}_i^p$ remains bounded, which in turn means that $\gamma_i(t) > 0$. Define $\gamma_i^* = \min_t \gamma_i(t)$. Hence, we have that $c \triangleq \gamma_i^* \frac{R_i - 2F}{R_i} > 0$. From the results in [25], since Dini derivative satisfies $D^+(V(\boldsymbol{\tau}_i)) \leq -cV(\boldsymbol{\tau}_i)^\beta$ for all $\boldsymbol{\tau}_i \in \mathbb{R}^2$, we obtain that $\boldsymbol{\tau}_L$ is finite-time stable, with the bound on the finite time of convergence given as $T_{1i} \leq \frac{V(\boldsymbol{\tau}_i(0))^{1-\beta}}{c(1-\beta)} = \frac{\|\boldsymbol{\tau}_i(0) - \boldsymbol{\tau}_L\|^{2(1-\beta)}}{2^{1-\beta}c(1-\beta)}$. Now, at $t = T_{1i}$, agent i has its $\boldsymbol{\tau}_i$ co-located with all the normal leaders' $\boldsymbol{\tau}$. This means that there can be at max F agents (i.e. the adversarial leaders) which are not co-located with the agent's $\boldsymbol{\tau}_i$. Hence, we obtain that $|\Omega_i(t)| \leq F$ for all $t \geq T_{1i}$. Therefore, by Theorem 1 agent i will stay at $\boldsymbol{\tau}_L$ for all future times. ■

Next we take the case of normal agents $i \in \mathcal{S}_2$:

Lemma 2: Consider a digraph $\mathcal{D}$ which is an RDAG with parameter $3F+1$, where $\mathcal{S}_0 = \mathcal{L}$ and $\mathcal{A}$ is an F-total set. Under the closed loop dynamics (3)-(7), the value $\boldsymbol{\tau}_i(t)$ for each normal agent $i \in \mathcal{S}_2$ converges to $\boldsymbol{\tau}_L$ in finite time T_{2i}.

Proof: For the worst case analysis, assume that all the agents in $\mathcal{R}_i(0)$ are from $\mathcal{S}_1$ and are located such that $(\boldsymbol{\tau}_j(0) - \boldsymbol{\tau}_i(0))^T(\boldsymbol{\tau}_L - \boldsymbol{\tau}_i(0)) < 0$ for each $j \in \mathcal{R}_i(0)$. This simply means that the agents in $\mathcal{R}_i$ at time $t = 0$ are located on one side of the agent while the leaders are on the other side. This is the worst case because this arrangement of in-neighbors would make agent i move away from the leaders, initially. Also, assume that $|\mathcal{R}_i^{\mathcal{A}}| = F$ and $|\mathcal{R}_i^{\mathcal{N}}| = R_i - F$, so that agent i has maximum number of adversarial in-neighbours. Consider the candidate Lyapunov function $V(\boldsymbol{\tau}_i(t)) = \frac{1}{2}\|\boldsymbol{\tau}_i(t) - \boldsymbol{\tau}_L\|^2$. Taking its upper right Dini derivative along the closed-loop trajectories of agent i, we have $D^+(V(\boldsymbol{\tau}_i)) = (\boldsymbol{\tau}_i - \boldsymbol{\tau}_L)^T \sum_{j \in \mathcal{R}_i} \gamma_i w_{ij}(\boldsymbol{\tau}_j - \boldsymbol{\tau}_i)\|\boldsymbol{\tau}_j - \boldsymbol{\tau}_i\|^{\alpha-1}$. Now, from the assumption on the initial locations of agents in $\mathcal{R}_i(t)$, we have that $D^+(V(\boldsymbol{\tau}_i(0))) = \gamma_i(0)\sum_{j \in \mathcal{R}_i} w_{ij}(\boldsymbol{\tau}_i - \boldsymbol{\tau}_L)^T(\boldsymbol{\tau}_j - \boldsymbol{\tau}_i)\|\boldsymbol{\tau}_j - \boldsymbol{\tau}_i\|^{\alpha-1} > 0$. Also, define $T_1 \triangleq \max_{l \in \mathcal{S}_1 \bigcap \mathcal{N}} T_{1l}$, i.e. T_1 is the maximum time after which each normal agent in $\mathcal{S}_1$ would achieve formation and have $\boldsymbol{\tau}_i = \boldsymbol{\tau}_L$. Hence, at time $t = T_1$, we have that:

$$\begin{aligned} D^+(V(\boldsymbol{\tau}_i)) &= \sum_{j \in \mathcal{R}_i^{\mathcal{N}}} \gamma_i w_{ij}(\boldsymbol{\tau}_i - \boldsymbol{\tau}_L)^T(\boldsymbol{\tau}_j - \boldsymbol{\tau}_i)\|\boldsymbol{\tau}_j - \boldsymbol{\tau}_i\|^{\alpha-1} \\ &+ \sum_{j \in \mathcal{R}_i^{\mathcal{A}}} \gamma_i w_{ij}(\boldsymbol{\tau}_i - \boldsymbol{\tau}_L)^T(\boldsymbol{\tau}_j - \boldsymbol{\tau}_i)\|\boldsymbol{\tau}_j - \boldsymbol{\tau}_i\|^{\alpha-1} \\ &= \gamma_i \frac{R_i - F}{R_i}(\boldsymbol{\tau}_i - \boldsymbol{\tau}_L)^T(\boldsymbol{\tau}_L - \boldsymbol{\tau}_i)\|\boldsymbol{\tau}_L - \boldsymbol{\tau}_i\|^{\alpha-1} \\ &+ \sum_{j \in \mathcal{R}_i^{\mathcal{A}}} \gamma_i w_{ij}(\boldsymbol{\tau}_i - \boldsymbol{\tau}_L)^T(\boldsymbol{\tau}_j - \boldsymbol{\tau}_i)\|\boldsymbol{\tau}_j - \boldsymbol{\tau}_i\|^{\alpha-1} \\ &\leq -\gamma_i \frac{R_i - F}{R_i}\|\boldsymbol{\tau}_L - \boldsymbol{\tau}_i\|^{1+\alpha} \\ &+ \gamma_i \sum_{j \in \mathcal{R}_i^{\mathcal{A}}} w_{ij}\|\boldsymbol{\tau}_i - \boldsymbol{\tau}_L\|\|\boldsymbol{\tau}_j - \boldsymbol{\tau}_i\|^{\alpha} \end{aligned}$$

Now, for all $j \in \mathcal{R}_i^{\mathcal{A}}$, the norm $\|\boldsymbol{\tau}_j(T_1) - \boldsymbol{\tau}_i(T_1)\| \leq \|\boldsymbol{\tau}_k(T_1) - \boldsymbol{\tau}_i(T_1)\|$ for some $k \in \mathcal{R}_i^{\mathcal{N}}$ otherwise, these adversarial agents would be filtered out. Using this and the fact that $\boldsymbol{\tau}_k(T_1) = \boldsymbol{\tau}_L$, we have that for all $t \geq T_1$:

$$D^+(V(\boldsymbol{\tau}_i(t))) \leq -\gamma_i \frac{R_i - F}{R_i} \|\boldsymbol{\tau}_L - \boldsymbol{\tau}_i\|^{1+\alpha} + \sum_{j \in \mathcal{R}_i^{\mathcal{A}}} \gamma_i w_{ij} \|\boldsymbol{\tau}_i - \boldsymbol{\tau}_L\| \|\boldsymbol{\tau}_L - \boldsymbol{\tau}_i\|^{\alpha} = -\gamma_i \frac{R_i - 2F}{R_i} \|\boldsymbol{\tau}_L - \boldsymbol{\tau}_i\|^{1+\alpha} < 0.$$

Since $D^+(V(\boldsymbol{\tau}_i))(0) > 0$ while $D^+(V(\boldsymbol{\tau}_i))(T_1) < 0$, and it is bounded above in the interval $(0, T_1)$, the increment in the value of $V(\boldsymbol{\tau}_i)$ is bounded in the interval. Hence, agent i would be at a finite distance away from the leaders at time T_1. This also implies that $\boldsymbol{u}_i^p(t)$ is bounded and hence $\gamma_i^* = \min_t \gamma_i(t) > 0$. Hence, we obtain that $D^+(V(\boldsymbol{\tau}_i)) \leq -cV(\boldsymbol{\tau}_i)^{\beta}$ where $c = \gamma_i^* \frac{R_i - 2F}{R_i} > 0$ and $\beta = \frac{1+\alpha}{2} < 1$. Hence, we have that $\boldsymbol{\tau}_i \to \boldsymbol{\tau}_L$ in finite time. Let $\boldsymbol{\tau}_i(T_1)$ be the position of agent at time instant T_1. Using the bound on finite time of convergence, we obtain that for $t \geq T_{2i}$, $\boldsymbol{\tau}_i(t) = \tau_L$ where

$$T_{2i} \leq T_1 + \frac{V(\boldsymbol{\tau}_i(T_1))^{1-\alpha}}{c(1-\alpha)} = T_1 + \frac{\|\boldsymbol{\tau}_i(T_1) - \boldsymbol{\tau}_L\|^{2(1-\beta)}}{2^{1-\beta} c(1-\beta)}$$

Since both T_1 and $\|\boldsymbol{\tau}_i(T_1) - \boldsymbol{\tau}_L\|$ are finite, $\alpha < 1$ and $c > 0$ we obtain that T_{2i} is also finite. Again, after time instant T_{2i}, agent i has its $\boldsymbol{\tau}_i$ co-located with all the normal in-neighbors' $\boldsymbol{\tau}$. This means that there can be at max F agents (i.e. the adversarial agents) which are not co-located with the agent's $\boldsymbol{\tau}_i$. Hence, we have that $|\Omega_i(t)| \leq F$ for all $t \geq T_{2i}$. Therefore, Theorem 1 implies that agent i will stay at $\boldsymbol{\tau}_L$ for all $t \geq T_{1i}$. ∎

We have shown that each normal agent $i \in \mathcal{S}_2$ will achieve the formation in finite time. Now we present the general case:

Theorem 2: Consider a digraph $\mathcal{D}$ which is an RDAG with parameter $3F + 1$, where $\mathcal{S}_0 = \mathcal{L}$ and $\mathcal{A}$ is an F-total set. Under the closed loop dynamics (3)-(7), $\boldsymbol{\tau}_i$ will converge to $\boldsymbol{\tau}_L$ in finite time for all normal agents $i \in \mathcal{N}$.

Proof: We have already shown that all the agents in $\mathcal{S}_1$ and $\mathcal{S}_2$ will achieve formation in finite time. Consider any agent $i \in S_3$. Since all the in-neighbors of agents in $\mathcal{S}_3$ are from $\bigcup_{i=0}^{2} \mathcal{S}_i$, after a finite time period all the agents in $\mathcal{V}_i \bigcap \mathcal{N}$ will satisfy $\boldsymbol{\tau}_i = \boldsymbol{\tau}_L$. Define $T_2 \triangleq \max_k T_{2k}$, where k belongs to the set of normal agents in $\mathcal{S}_1$. After the time instant $t = T_2$, the Lyapunov candidate $V(\boldsymbol{\tau}_i) = \frac{1}{2}\|\boldsymbol{\tau}_i - \boldsymbol{\tau}_L\|^2$ and its Dini derivative will satisfy the conditions similar to Lemma 2. Hence, we have that all the normal agents in $\mathcal{S}_3$ will achieve formation in finite time. This time can be bounded as $T_{3i} \leq T_2 + \frac{\|\boldsymbol{\tau}_i(T_2) - \boldsymbol{\tau}_L\|^{1-\alpha}}{c(1-\alpha)}$ for each $i \in \mathcal{S}_3$. This argument can be used recursively to show that each normal agent in $\bigcup_{l=1}^{p} \mathcal{S}_l$ will achieve formation in finite time. Defining T_l as the maximum time by which all the normal agents in set $\mathcal{S}_l$ will achieve the formation, one can establish the following relation for $l \geq 1$:

$$T_{l+1} \leq T_l + \max_{i \in \mathcal{S}_{l+1}} \frac{\|\boldsymbol{\tau}_i(T_l) - \boldsymbol{\tau}_L\|^{2(1-\beta)}}{2^{1-\beta} c(1-\beta)},$$

where $T_1 \leq \max_{i \in \mathcal{S}_1} \frac{\|\boldsymbol{\tau}_i(0) - \boldsymbol{\tau}_L\|^{2(1-\beta)}}{2^{1-\beta} c(1-\beta)}$. Since T_l and $\|\boldsymbol{\tau}_i(T_l) - \boldsymbol{\tau}_L\|$ both are finite $\forall\ l \geq 1$, we have $T_{l+1} < \infty$. ∎

Hence, we have shown that under the effect of our protocol, each normal agent i would achieve formation in finite time, despite adversarial agents. In the next section, we show that our filtering mechanism can be used for the case of discrete time systems as well.

IV. Discrete-time System

A. Filtering Algorithm and Control Law

At each time step t, each agent $i \in \mathcal{N}$ applies the following algorithm:

Algorithm 2 Discrete-Time Filtering

procedure UpdateFilteredList
  Calculate $\tau_{ij} = \|\boldsymbol{\tau}_j - \boldsymbol{\tau}_i\| \ \ \forall j \in \mathcal{V}_i$
  Sort τ_{ij} values such that $\tau_{ij_1} \geq \ldots \geq \tau_{ij_{|\mathcal{V}_i|}}$
  $\mathcal{R}_i[t] \leftarrow \{j : \tau_{ij} \in \{\tau_{ij_{F+1}}, \ldots, \tau_{ij_{|\mathcal{V}_i|}}\}\}$

The discrete time system dynamics are given as

$$\boldsymbol{\tau}_i[t+1] = \boldsymbol{p}_i[t+1] - \boldsymbol{\xi}_i = \boldsymbol{p}_i[t] + \boldsymbol{u}_i[t] - \boldsymbol{\xi}_i = \boldsymbol{\tau}_i[t] + \boldsymbol{u}_i[t] \tag{8}$$

The input of each agent i is bounded above by $u_M > 0$, i.e. $\|\boldsymbol{u}_i[t]\| \leq u_M$ for all $t \geq 0$. Under this constraint, the saturation function is given as

$$\sigma_i[t] = \min\{\|\boldsymbol{u}_i^p[t]\|, u_M\}, \tag{9}$$

$$\boldsymbol{u}_i^p[t] = \sum_{j \in \mathcal{R}_i[t]} w_{ij}[t](\boldsymbol{\tau}_j[t] - \boldsymbol{\tau}_i[t]). \tag{10}$$

To simplify the notation, define $\gamma_i[t] = \frac{\sigma_i[t]}{\|\boldsymbol{u}_i^p[t]\|}$. We define the control law $\boldsymbol{u}_i[t]$ as

$$\boldsymbol{u}_i[t] = \gamma_i[t] \sum_{j \in \mathcal{R}_i[t]} w_{ij}[t](\boldsymbol{\tau}_j[t] - \boldsymbol{\tau}_i[t]), \tag{11}$$

where for all time steps t and for all $i \in \mathcal{N}$, $w_{ij}[t] > 0$ and $\sum_{j \in \mathcal{R}_i[t]} w_{ij}[t] = 1$. For simplicity, we choose $w_{ij}[t] = \frac{1}{R_i}$. We point out that $0 < \gamma_i[t] \leq 1$. In the following subsection, we prove that under the effect of the control law (11) and Algorithm 2, normal behaving agents in the discrete time setting are also guaranteed to achieve formation despite the presence of adversarial agents.

B. Convergence Analysis

For our analysis, we need the following result:

Lemma 3: Let $b[k] = kc^k b[0],\ k \in \mathbb{Z}_{\geq 0}$ be a series where $b[0] > 0$ and $0 < c < 1$. Then there exist positive constants α, β with $c < \beta < 1$ such that $\forall k \in \mathbb{Z}_{\geq 0}$,

$$b[k] = kc^k b[0] \leq \alpha \beta^k. \tag{12}$$

Proof: It can be readily verified that for any $c < \beta < 1$ and $\alpha \geq \frac{b[0]}{e \log \frac{\beta}{c}}$, the inequality (12) holds for all $k \geq 0$. ∎

First, consider the normal agents in the set $\mathcal{S}_1$:

Lemma 4: Consider a digraph $\mathcal{D}$ which is an RDAG with parameter $3F + 1$, where $\mathcal{S}_0 = \mathcal{L}$ and $\mathcal{A}$ is an F-total set.

For every normal agent $i \in \mathcal{S}_1$, $\|\boldsymbol{\tau}_i[t] - \boldsymbol{\tau}_L\|$ converges to zero exponentially.

Proof: For the worst case, assume there are F adversarial agents. Consider any normal agent $i \in \mathcal{S}_1$. Since all of its in-neighbours are from $\mathcal{S}_0$, we have that $\mathcal{V}_i \subset \mathcal{L}$ and for all $k \in \mathcal{V}_i \bigcap \mathcal{N}$, $\boldsymbol{\tau}_k = \boldsymbol{\tau}_L$. By definition of an RDAG, $|\mathcal{V}_i| \geq 3F+1$ which implies $R_i \geq 2F+1$ and $|\mathcal{R}_i^{\mathcal{N}}| \geq F+1$. For the worst case, suppose that $\|\boldsymbol{\tau}_i[t]-\boldsymbol{\tau}_j[t]\| \leq \|\boldsymbol{\tau}_i[t]-\boldsymbol{\tau}_L\|$ $\forall j \in \mathcal{A}_i$ so that none of the adversarial agents are filtered out. This implies that $|\mathcal{R}_i^{\mathcal{A}}| = F$ and $|\mathcal{R}_i^{\mathcal{N}}| = R_i - F$. From the closed loop dynamics, we obtain:

$$\boldsymbol{\tau}_i[t+1] - \boldsymbol{\tau}_L = \boldsymbol{\tau}_i[t] + \sum_{j \in \mathcal{R}_i} \gamma_i w_{ij}(\boldsymbol{\tau}_j[t] - \boldsymbol{\tau}_i[t]) - \boldsymbol{\tau}_L.$$

Noting that $\mathcal{R}_i \subset \mathcal{L}$, after some manipulation we obtain:

$$\begin{aligned}\boldsymbol{\tau}_i[t+1] - \boldsymbol{\tau}_L = (1 - \gamma_i \frac{R_i - F}{R_i})(\boldsymbol{\tau}_i[t] - \boldsymbol{\tau}_L) \\ + \sum_{j \in \mathcal{R}_i^{\mathcal{A}}} \gamma_i w_{ij}(\boldsymbol{\tau}_j[t] - \boldsymbol{\tau}_i[t]). \qquad (13)\end{aligned}$$

Since $\|\boldsymbol{\tau}_i[t] - \boldsymbol{\tau}_j[t]\| \leq \|\boldsymbol{\tau}_i[t] - \boldsymbol{\tau}_L[t]\|$ for all $j \in \mathcal{R}_i^{\mathcal{A}}$, we have $\|\sum_{j \in \mathcal{R}_i^{\mathcal{A}}} w_{ij}(\boldsymbol{\tau}_j[t] - \boldsymbol{\tau}_i[t])\| \leq \frac{F}{|\mathcal{R}_i|}\|\boldsymbol{\tau}_i[t] - \boldsymbol{\tau}_L\|$. Hence, we obtain the bound on $\|\boldsymbol{\tau}_i[t+1] - \boldsymbol{\tau}_L\|$ as:

$$\|\boldsymbol{\tau}_i[t+1] - \boldsymbol{\tau}_L\| \leq (1 - \gamma_i \frac{R_i - 2F}{R_i})\|\boldsymbol{\tau}_i[t] - \boldsymbol{\tau}_L\|. \quad (14)$$

Let $\gamma_i^* = \min_k \gamma_i[k] > 0$. Since $1 - \gamma_i \frac{R_i - 2F}{R_i} \leq 1 - \gamma_i^* \frac{R_i-2F}{R_i} < 1$, define $c = 1 - \gamma_i^* \frac{R_i-2F}{R_i}$, so that we have $\|\boldsymbol{\tau}_i[t+1] - \boldsymbol{\tau}_L\| \leq c\|\boldsymbol{\tau}_i[t] - \boldsymbol{\tau}_L\|$, i.e. $\|\boldsymbol{\tau}_i[t] - \boldsymbol{\tau}_L\|$ is an exponentially converging sequence. ■

For $i \in \mathcal{S}_p$ where $p \geq 2$, we know that there are at most F adversarial agents in $\mathcal{R}_i$. Note that by definition of the network RDAG, all agents in $\mathcal{R}_i$ are from $\bigcup_{j=0}^{p-1} \mathcal{S}_j$. For the worst-case analysis, we assume there are F adversarial agents in $\mathcal{R}_i$ and all the normal agents in $\mathcal{R}_i$ are from $\mathcal{S}_{p-1}$. From the closed-loop dynamics of the agent i, we have:

$$\boldsymbol{\tau}_i[t+1] - \boldsymbol{\tau}_L = \boldsymbol{\tau}_i[t] + \sum_{j \in \mathcal{R}_i} \gamma_i w_{ij}(\boldsymbol{\tau}_j[t] - \boldsymbol{\tau}_i[t]) - \boldsymbol{\tau}_L,$$

which after some manipulation gives:

$$\begin{aligned}\boldsymbol{\tau}_i[t+1] - \boldsymbol{\tau}_L = (1 - \gamma_i \frac{R_i - F}{R_i})(\boldsymbol{\tau}_i[t] - \boldsymbol{\tau}_L) \\ + \sum_{j \in \mathcal{R}_i^{\mathcal{N}}} \gamma_i w_{ij}(\boldsymbol{\tau}_j[t] - \boldsymbol{\tau}_L) + \sum_{j \in \mathcal{R}_i^{\mathcal{A}}} \gamma_i w_{ij}(\boldsymbol{\tau}_j[t] - \boldsymbol{\tau}_i[t]). \\ (15)\end{aligned}$$

Using the same logic as in Lemma 4, we assume for the worst case that $\forall j \in \mathcal{A}_i$, $\|\boldsymbol{\tau}_i[t] - \boldsymbol{\tau}_j[t]\| \leq \|\boldsymbol{\tau}_i[t] - \boldsymbol{\tau}_k[t]\|$ for some $k \in \mathcal{R}_i^{\mathcal{N}}$. Using this, the fact that $|\mathcal{R}_i^{\mathcal{A}}| = F$, we obtain:

$$\|\sum_{j \in \mathcal{R}_i^{\mathcal{N}}} w_{ij}(\boldsymbol{\tau}_j[t] - \boldsymbol{\tau}_L)\| \leq \frac{|\mathcal{R}_i| - F}{|\mathcal{R}_i|}\|\boldsymbol{\tau}_k[t] - \boldsymbol{\tau}_L\|,$$

$$\|\sum_{j \in \mathcal{R}_i^{\mathcal{A}}} w_{ij}(\boldsymbol{\tau}_j[t] - \boldsymbol{\tau}_i[t])\| \leq \frac{F}{|\mathcal{R}_i|}\|\boldsymbol{\tau}_k[t] - \boldsymbol{\tau}_i\|.$$

We can bound $\|\boldsymbol{\tau}_k - \boldsymbol{\tau}_i\| \leq \|\boldsymbol{\tau}_k - \boldsymbol{\tau}_L\| + \|\boldsymbol{\tau}_i - \boldsymbol{\tau}_L\|$ to obtain:

$$\|\boldsymbol{\tau}_i[t+1] - \boldsymbol{\tau}_L\| \leq c\|\boldsymbol{\tau}_i[t] - \boldsymbol{\tau}_L\| + \|\boldsymbol{\tau}_k - \boldsymbol{\tau}_L\|, \qquad (16)$$

where $c = 1 - \gamma_i^* \frac{R_i - 2F}{R_i} < 1$ where γ_i^* is defined as in Lemma 4. Inequality (16) is true for every normal agent in $\mathcal{S}_p$ with $p \geq 2$. Using this observation, we next consider the case of agents in set $\mathcal{S}_2$:

Lemma 5: Consider a digraph $\mathcal{D}$ which is an RDAG with parameter $3F+1$, where $\mathcal{S}_0 = \mathcal{L}$ and $\mathcal{A}$ is an F-total set. For every normal agent $i \in \mathcal{S}_2$, $\|\boldsymbol{\tau}_i[t] - \boldsymbol{\tau}_L\|$ converges to zero exponentially.

Proof: Define $a[t] = \|\boldsymbol{\tau}_i[t] - \boldsymbol{\tau}_L\|$, $b_k[t] = \|\boldsymbol{\tau}_k[t] - \boldsymbol{\tau}_L\|$ so that (16) can be written as $a[t+1] \leq ca[t] + b_k[t]$:

$$a[t+1] \leq c^{t+1}a[0] + \sum_{i=0}^{t} c^{t-i} b_k[i]. \qquad (17)$$

Now, $b_k[i]$ represents the norm $\|\boldsymbol{\tau}_k[i] - \boldsymbol{\tau}_L\|$ of a normal agent $k \in \mathcal{S}_1$, which can be bounded as $b_k[i] \leq c_k^i b_k[0]$ as per (14) where $c_k = 1 - \gamma_k^* \frac{R_k - 2F}{R_k} < 1$. For the sake of brevity, let $a_0 = a[0]$, $b_{k0} = b_k[0]$. Using this, we obtain:

$$a[t+1] \leq c^{t+1}a_0 + \sum_{i=0}^{t} c^{t-i} c_k^i b_{k0}$$

Define $b_0^* = \max_{k \in \mathcal{R}_i^{\mathcal{N}}} b_{k0}$, $c^* = \max_{k \in \mathcal{R}_i^{\mathcal{N}}} c_k$, and $\tilde{c} = \max\{c, c^*\}$, so that

$$a[t+1] \leq \tilde{c}^{t+1}a_0 + \sum_{i=0}^{t} \tilde{c}^t b_0^* = \tilde{c}^{t+1}a_0 + (t+1)\tilde{c}^t b_0^*.$$

Using this and Lemma 3, i.e., $k\tilde{c}^t b_0^* \leq \alpha\beta^t$, we have that:

$$a[t+1] \leq \tilde{c}^t(ca_0 + b_0^*) + t\tilde{c}^t b_0^* \leq \tilde{c}^t(ca_0 + b_0^*) + \alpha\beta^t,$$

where $\alpha > 0$ and $\tilde{c} < \beta < 1$. Now, since $\tilde{c} < \beta$, we have:

$$a[t+1] \leq \tilde{c}^t(ca_0 + b[0]) + \alpha\beta^t \leq \beta^t(ca_0 + b[0] + \alpha).$$

As $\beta < 1$, a_t converges to zero exponentially, i.e., for a normal agent $i \in \mathcal{S}_2$, $\|\boldsymbol{\tau}_i[t] - \boldsymbol{\tau}_L\|$ converges to zero exponentially. ■

Note that this result can be interpreted as follows: $\|\boldsymbol{\tau}_i - \boldsymbol{\tau}_L\|$ for $i \in \mathcal{N}$ converges to zero exponentially if $\|\boldsymbol{\tau}_j - \boldsymbol{\tau}_L\|$ converges to zero exponentially for all its normal in-neighbours $j \in \mathcal{R}_i \bigcap \mathcal{N}$. Using this, we can state the following result for all normal behaving agents:

Theorem 3: Consider a digraph $\mathcal{D}$ which is an RDAG with parameter $3F+1$, where $\mathcal{S}_0 = \mathcal{L}$ and $\mathcal{A}$ is an F-total set. Under the closed loop dynamics (8)-(11), $\|\boldsymbol{\tau}_i[t] - \boldsymbol{\tau}_L\|$ converges to zero exponentially for all agents $i \in \mathcal{N}$.

Proof: We have proven this result for agents in $\mathcal{S}_1$ and $\mathcal{S}_2$ in Lemmas 4 and 5. We now consider any node $i \in \mathcal{S}_p$ for arbitrary p. Observe that every agent $i \in \mathcal{S}_p$ satisfies the equation (17), where $a[t]$ represents the norm $\|\boldsymbol{\tau}_i[t] - \boldsymbol{\tau}_L\|$ and $b_j[t] = \|\boldsymbol{\tau}_j[t] - \boldsymbol{\tau}_L\|$, where $j \in \bigcup_{l=0}^{p-1} \mathcal{S}_l$. From Lemma 5, we have that $\|\boldsymbol{\tau}_i - \boldsymbol{\tau}_L\|$ for normal agents in $\mathcal{S}_2$ converges exponentially to zero. Hence, it follows that for each normal agent in $\mathcal{S}_3$, $\|\boldsymbol{\tau}_i - \boldsymbol{\tau}_L\|$ converges to zero exponentially since all of its normal behaving agents are from the set $\bigcup_{l=0}^{2} \mathcal{S}_l$.

Repeating this logic shows that for each normal agent $i \in \mathcal{S}_p$, $\|\boldsymbol{\tau}_i - \boldsymbol{\tau}_L\|$ converges exponentially to zero for each $p \geq 1$. ∎

V. Simulation

We consider an RDAG of 80 agents with parameter $r = 16$ and $F = 5$. There are 5 sub-levels, $\mathcal{S}_l$ with $|\mathcal{S}_l| = 16$ for $l \in \{0, 1, 2, 3, 4\}$. The set $\mathcal{S}_0$ is composed entirely of agents designated to behave as leaders. In the simulation, $\mathcal{A}$ is a 5-local model with 5 agents in each of the levels $\mathcal{S}_l$ becoming adversarial (including in $\mathcal{S}_0$). The simulation treats a worst-case scenario in the sense that each agent $i \in \mathcal{S}_l$, $l \geq 1$, has in-neighbours only in $\mathcal{S}_{l-1}$ and no leader in-neighbors. The agents have states in $\mathbb{R}^2$. The vector $\boldsymbol{\xi}$ specifies the formation as points on circle of radius 10 m centered at $\begin{bmatrix} 0 & 10 \end{bmatrix}^T$. The vectors $\boldsymbol{p}_i(0)$, $i \in \mathcal{L}$ are chosen such that $\boldsymbol{\tau}_i(0)$ is at the origin for all $i \in \mathcal{L}$. The vectors $\boldsymbol{p}_j(0)$ for all other agents $j \in (\mathcal{V}_i \backslash \mathcal{L})$ are initialized such that their $\boldsymbol{\tau}_j(0)$ values are randomly initialized values. We choose the maximum allowed speed of the agents as $u_M = 1$. These conditions are used for both the continuous and discrete time simulations. For the continuous time case, the control parameter α is chosen as $\alpha = 0.8$.

Figure 1 shows a plot of $\|\boldsymbol{\tau}_i(t) - \boldsymbol{\tau}_L\|$ versus time for a subset of the normal agents. It is clear that all the normal agents converge to the point where their $\boldsymbol{\tau}$ values are same as those of leader in finite time. Figure 2 shows the path $\boldsymbol{p}_i(t) = \begin{bmatrix} x_i(t) & y_i(t) \end{bmatrix}^T$ of all the agents and a subset of the adversarial agents. The paths of the adversarial agents are chosen to as random walks and are not shown in the figure. In Figure 2 and Figure 5, it can be noted that while some normal agents (belonging to set $\mathcal{S}_1$ move directly towards their desired locations, other normal agents first move away from their desired locations. This is in agreement with our analysis; malicious agents are able to exert a bounded influence on normal agents in $\mathcal{S}_l$, $l \geq 2$ which do not have any leaders as in-neighbors, while convergence is still guaranteed in a finite time period.

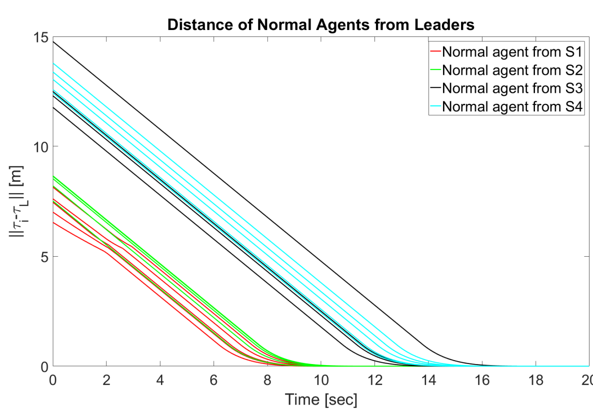

Fig. 1. Norm $\|\boldsymbol{\tau}_i(t) - \boldsymbol{\tau}_L\|$ of a subset of the normal agents in the continuous time case. For sake of clarity, only a few normal nodes from each set $\mathcal{S}_p$ are shown.

For the case of discrete system, Figure 4 shows the variation of $\|\boldsymbol{\tau}_i[k] - \boldsymbol{\tau}_L\|$ with number of steps. Figure 5 shows the path $\boldsymbol{p}_i[k] = \begin{bmatrix} x_i[k] & y_i[k] \end{bmatrix}^T$ of the agents. From both the figures, it is clear that despite the 5-local adversarial model, each normal agent achieves the desired formation.

From Figures 1 and 4, it can be seen that agents in $\mathcal{S}_l$ converge before agents in $\mathcal{S}_{l+1}$, which is consistent with our analysis for this particular worst-case scenario.

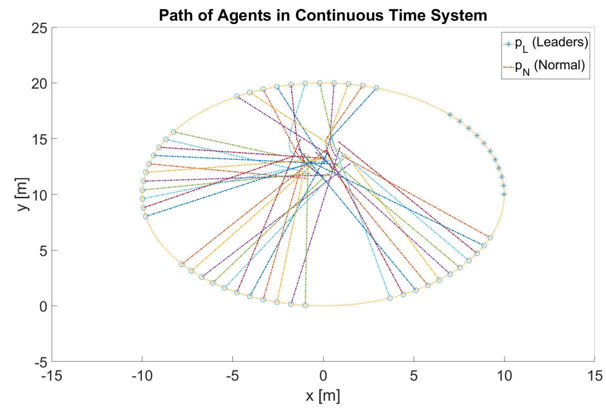

Fig. 2. Path of the agents in the continuous time case. All normal and misbehaving agents start from the centre of the circle marked by red dots. The leaders are denoted by the star points $\mathrm{p_L}$ and the non-adversarial agents are denoted by $\mathrm{p_N}$.

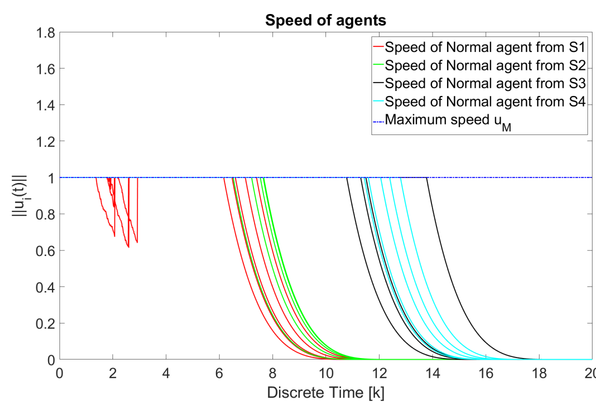

Fig. 3. Norm $\|\boldsymbol{u}_i(t)\|$ of a subset of the normal agents, demonstrating that their input magnitudes never exceed the bound $u_M = 1$. The rest of the network is not shown for sake of clarity.

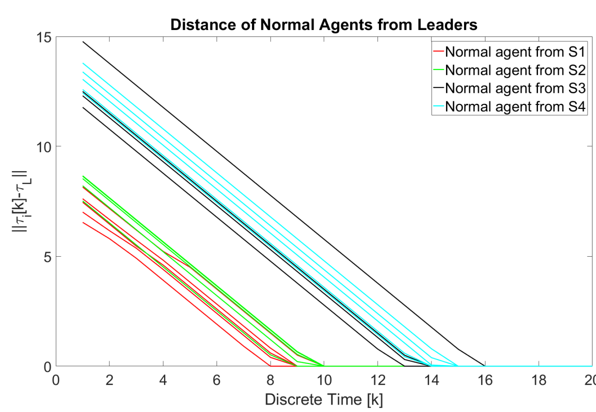

Fig. 4. Norm $\|\boldsymbol{\tau}_i[k] - \boldsymbol{\tau}_L\|$ of a subset of the normal agents in the continuous time case. For sake of clarity, only a few normal nodes from each set $\mathcal{S}_p$ are shown.

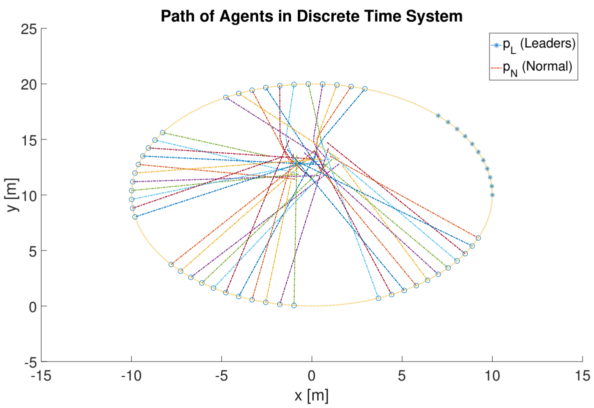

Fig. 5. Path of the agents in the discrete time case. The leaders are denoted by the star points $\mathrm{p_L}$ and the non-adversarial agents are denoted by $\mathrm{p_N}$.

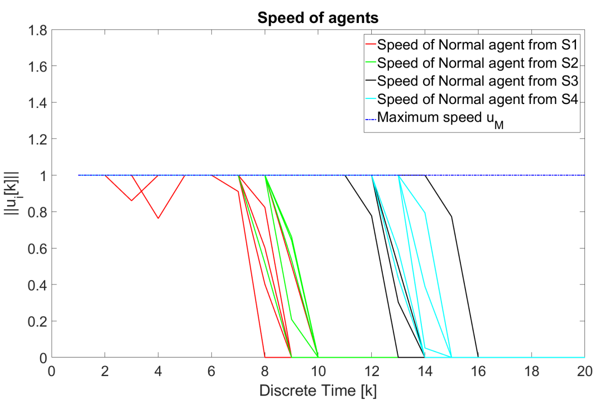

Fig. 6. Norm $\|\boldsymbol{u}_i[k]\|$ of a subset of the normal agents in the discrete time case. Again, the magnitude of each agents' control input never exceeds the bound $u_M = 1$ and goes to zero as the agents converge to formation.

VI. Conclusion

In this paper we introduced a novel continuous time resilient controller which guarantees that normally behaving agents can converge to a formation with respect to a set of leaders in the presence of adversarial agents. We proved that even with bounded inputs, the controller guarantees convergence in finite-time. In addition, we also applied our filtering mechanism to a discrete-time system and showed that it guarantees exponential convergence of agents to formation in the presence of adversaries under bounded inputs. Future work in this area will include further analysis of establishing safety among the normal agents and extending our results to time-varying graphs.

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