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Strong Invariance Using Control Barrier Functions: A Clarke Tangent Cone Approach*

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***Abstract*— Many control applications require that a system be constrained to a particular set of states, often termed as safe set. A practical and flexible method for rendering safe sets forward-invariant involves computing control input using Control Barrier Functions and Quadratic Programming methods. Many prior results however require the resulting control input to be continuous, which requires strong assumptions or can be difficult to demonstrate theoretically. In this paper we use differential inclusion methods to show that simultaneously rendering multiple sets invariant can be accomplished using a discontinuous control input. We present an optimization formulation which computes such control inputs and which can be posed in multiple forms, including a feasibility problem, a linear program, or a quadratic program. In addition, we discuss conditions under which the optimization problem is feasible and show that any feasible solution of the considered optimization problem which is measurable renders the multiple safe sets forward invariant.**

I. Introduction

Safety considerations such as maintaining a safe distance from static or dynamic obstacles for systems like robots, unmanned aerial vehicles, and autonomous cars is a critical concern in modern control theory. Safety requirements and other system objectives, such as confining the system trajectories to remain in a desired operating set, can be modelled as set invariance constraints where the objective is to guarantee that state trajectories remain within specified subsets of the state space under the closed-loop dynamics. Among other approaches, control barrier functions (CBFs) have been studied by many researchers to establish forward invariance of safe sets, thereby guaranteeing safety and other system objectives are achieved [1]–[3]. More recently, in [2], [4], conditions using zeroing control barrier functions (ZCBF) are presented to ensure forward invariance of a desired set. Other authors have used CBFs to design control input using closed-form expressions that resemble Sontag's formula, e.g., [1].

For certain classes of nonlinear systems it can be difficult in general to find closed-form expressions for control inputs that render particular safe sets invariant. The authors in [2], [4]–[7] have explored online optimization methods of utilizing CBFs in control design, where typically, a quadratic program (QP) is set up to compute the control input at every point in the state space. In these works the CBF inequalities take form of the linear constraints in the QP. Since the QP needs to be solved pointwise in the state space, it becomes a parameteric optimization problem where the state variable acts as a parameter. The authors in [8] studied parameteric convex optimization problems, and showed that the solution of the optimization is continuously differentiable if the objective function and the constraints functions are twice continuously differentiable, and strict complementary slackness holds. These conditions are relaxed in [9], where only continuity of these functions is assumed to guarantee that the solution of the parameteric convex optimization problem is a continuous function of the parameter.

In the particular context of control design using QPs, the authors in [4] showed Lipschitz continuity of the solution of a QP under the assumption that the objective function and the functions defining the constraints in the QP are locally Lipschitz continuous, in the absence of control input constraints (see also [10]). Under similar assumptions, the authors in [6] show that the solution of the QP is guaranteed to be Lipschitz continuous (in the absence of input constraints) if the CBF constraints are inactive, i.e., the constraints are satisfied with strict inequality at the optimal solution z^*. However demonstrating the Lipschitz continuity of the optimal solution for more general conditions can be a nontrivial task.

The topic of guaranteeing set invariance under a possibly discontinuous control input has been studied for decades [11]–[14]. Only somewhat recently has some of this theory been applied to set invariance using CBFs [6], [7], [15]. In [7] the forward-invariance of multiple safety sets is considered under a discontinuous control input. Their methods involve incorporating multiple CBFs into a single nonsmooth function and then utilizing the generalized gradient and set-valued Lie derivative to demonstrate forward invariance. This methodology requires computationally tracking the notion of almost active gradients and considering set-valued inner products to generate the required control inputs.

This paper presents a different approach to guaranteeing strong invariance of multiple composed sets as compared to prior literature. The first contribution of this paper is to guarantee the simultaneous forward invariance of multiple subsets of the state space using CBFs and incorporating control input constraints. Unlike prior work, we approach the problem using the notions of Clarke tangent cones and transversality. We demonstrate that a constrained control input simultaneously rendering these subsets invariant can

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be generated by simply solving a feasibility problem with compact linear constraints. The control input is only required to be Lebesgue measurable and is not required to be continuous. In contrast to [6], [7], we derive a set-valued map of feasible controls rendering the composed sets strongly invariant which is not only upper semicontinuous but also locally Lipschitz on a specified domain.

Our second contribution is formulating a general convex optimization problem which computes control inputs that simultaneously render multiple subsets invariant. This optimization problem takes the form of a feasibility problem, with special cases being a Linear Program (LP) and QP. In contrast to [6], [7] we show that under certain assumptions the optimization problem is feasible, even in the presence of control input constraints. The feasibility of the optimization problem is proven sufficient to guarantee forward invariance of multiple safe sets without requiring a continuity property of its solution as a function of the system states.

Some proofs are omitted from this paper for brevity, but are available online in the extended version [16].

II. Notation

For a closed set $S \subset \mathbb{R}^n$, ∂S denotes its boundary, $\text{int}(S)$, the interior, $\overline{\text{co}}(S)$ its closed convex hull of S and $S^\circ = \{y \in \mathbb{R}^n : \langle y, x\rangle \leq 0 \ \forall x \in S\}$ its polar cone. The function d_S is defined as $d_S(x) = \inf\{\|x - s\| : s \in S\}$. The power set of S is denoted $\mathcal{P}(S)$.

Given subsets $U \subset \mathbb{R}^{n\times m}$, $V \subset \mathbb{R}^{m\times p}$ the set-valued matrix product is defined as $UV = \{AB : A \in U, \ B \in V\} \subset \mathbb{R}^{n\times p}$. The Minkowski sum is denoted $U + V = \{A + B : A \in U, \ B \in V\}$. The Minkowski difference is $U_1 - U_2 = (U_1^c + U_2)^c$, where $U^c = \mathbb{R}^{n\times m}\backslash U$ denotes the set complement. Given a function $f : \mathbb{R}^{n\times m} \to \mathbb{R}^{m\times p}$, we denote $f(U) = \{f(A) : A \in U\}$. The norm $\|\cdot\|$ in this paper refers to any sub-multiplicative matrix norm, i.e. $\|AB\| \leq \|A\|\|B\|$. The open unit ball on a vector space $\mathbb{R}^{n\times m}$ centered at the origin is denoted $B^{n\times m}(0, 1)$. The closed unit ball is denoted $\bar{B}^{n\times m}(0, 1) = \overline{\text{co}}(B^{n\times m}(0, 1))$. The unit ball will be denoted as simply $B(0, 1)$ when the dimensions are clear from the context.

We use $h \in \mathcal{C}^{1,1}_{loc}$ to denote a continuously differentiable function, whose gradient ∇h is locally Lipschitz continuous. The Lie derivative of a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ along a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is denoted $L_f h(x) \triangleq \frac{\partial h}{\partial x} f(x)$. A function $g : \mathbb{R}^n \to \mathbb{R}$ is *locally bounded* on a set $D \subseteq \mathbb{R}^n$ if for all $x \in D$ there exists a neighborhood of x denoted $U(x)$ and a constant $M \in \mathbb{R}$ such that $\|g(z)\| \leq M$ for all $z \in U(x)$.

III. Problem Formulation

Consider the control affine system

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad u(t) \in \mathcal{U} \subset \mathbb{R}^m. \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are assumed to be locally Lipschitz on $\mathbb{R}^n$. Without loss of generality, we let $t_0 = 0$. The set $\mathcal{U} \subset \mathbb{R}^m$ represents the set of *feasible controls* for the system.

Assumption 1. *The set $\mathcal{U}$ is a compact, convex polytope with $\text{int}(\mathcal{U}) \neq \emptyset$ which has the form*

$$\begin{aligned} \mathcal{U} &= \{u \in \mathbb{R}^m : A_u u \leq b_u\}, \\ A_u &\in \mathbb{R}^{p\times m}, \ b_u \in \mathbb{R}^{p\times 1} \end{aligned} \tag{2}$$

where A_u, b_u are constant.

Constraints of this form are common in prior literature, e.g. [4], [17]–[19]. It is desired for the system trajectories $x(t)$ to remain within the intersection of multiple subsets $S_i \subseteq \mathbb{R}^n$, $i = 1, \ldots, N_h$ of the state space for all forward time. Each set S_i is defined as the sublevel[1] set of a continuous function $h_i : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\begin{aligned} S_i &= \{x \in \mathbb{R}^n : h_i(x) \leq 0\}, \\ \text{int}(S_i) &= \{x \in \mathbb{R}^n : h_i(x) < 0\}, \\ \partial S_i &= \{x \in \mathbb{R}^n : h_i(x) = 0\}. \end{aligned} \tag{3}$$

To characterize the properties of each h_i, we will use the notion of *strict* CBFs:

Definition 1. *The continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ is called a* ***strict*** *CBF for the set $S \subset \mathbb{R}^n$ defined as $S = \{x \mid h(x) \leq 0\}$ if the following holds:*

$$\inf_{u\in\mathcal{U}} [L_f h(x) + L_g h(x)u] < 0 \ \forall x \in \partial S, \tag{4}$$

Note that the authors in [4] call h a CBF if (4) holds with a non-strict inequality. Although the condition in (4) may be stronger than necessary when $u(t)$ is guaranteed to be continuous, the property in (4) will be useful when guaranteeing set invariance without a continuous control input. It is worth noting that this condition has been used in prior literature, e.g. [6, Prop. 3], to guarantee forward-invariance using a control input defined as a solution of a QP. We make the following assumptions on the safe sets:

Assumption 2. *Each h_i from (3) satisfies $h_i \in \mathcal{C}^{1,1}_{loc}$, and is a strict CBF.*

Assumption 3. *Each set S_i is compact.*

Remark 1. *Assumption 3 will aid the analysis of preventing finite escape time behavior of solutions. It is possible to encode such requirements using CLFs in the optimization framework [4], [5].*

The objective of this paper is to render the intersection $\bigcap_{i=1}^{N_h} S_i$ forward invariant for the closed-loop trajectories of (1) under a possibly discontinuous control input $u(\cdot)$.

Problem 1. *Given the sets $S_1, \ldots, S_{N_h}$ defined by (3) and system dynamics defined by (1), compute a possibly discontinuous control input u which renders $\bigcap_{i=1}^{N_h} S_i$ forward invariant. More specifically, compute a $u(t)$ such that any trajectory $x(t)$ of (1) with $x(0) \in \bigcap_{i=1}^{N_h} S_i$ satisfies*

$$x(t) \in \bigcap_{i=1}^{N_h} S_i, \ \forall t \geq 0. \tag{5}$$

[1] It is also common in prior literature to define each S_i in terms of *superlevel* sets, e.g. [2].

In particular, we seek to solve Problem 1 when $u(\cdot)$ may be Lebesgue measurable, but not necessarily continuous.

A. Differential Inclusion Theory and Strong Invariance

To guarantee forward invariance under a discontinuous $u(\cdot)$, this paper uses results from differential inclusion theory. A more detailed overview is given in [16, Appendix]. Consider a differential inclusion

$$\dot{x}(t) \in F(x(t)), \tag{6}$$

where $F : \mathcal{D} \subseteq \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$. Note that single-valued functions, e.g. $F(x(t)) = \{f(x(t))\}$ are a special case of differential inclusions. Existence of solutions to (6) are guaranteed by the following conditions which will be used in this paper.

Assumption 4 (Standing Hypotheses [13, Ch 4])**.** *The following conditions hold:*

a) For every $x \in \mathcal{D} \subseteq \mathbb{R}^n$, $F(x)$ is nonempty, compact, and convex;
b) $x \mapsto F(x)$ is upper semicontinuous;
c) $F(x)$ is locally bounded; i.e. for all $x \in \mathbb{R}^n$ there exist $\epsilon, m > 0$ such that $\|z\| \leq m$ for all $z \in F(y)$, for all $y \in B(x, \epsilon)$.

Since solutions to (6) may not necessarily be unique, forward invariance of a set $S \subset \mathbb{R}^n$ under system dynamics (6) is described by the concept of *strong invariance*.

Definition 2 ([11])**.** *Consider a differential inclusion (6) and let $S \subset \mathbb{R}^n$. The system pair (S, F) is said to be strongly invariant if* **all** *trajectories of the system $x(\cdot)$ with $x(0) \in S$ satisfy $x(t) \in S$ for all $t \geq 0$.*

Central to this paper's analysis of strong invariance is the Clarke tangent cone to S at x, denoted $T_S(x)$, which is defined next.

Definition 3 ([11])**.** *The Clarke tangent cone of the set S at x, denoted $T_S(x)$, is defined as*

$$T_S(x) = \left\{ v \in \mathbb{R}^n : \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{d_S(y + tv) - d_S(y)}{t} \leq 0 \right\}. \tag{7}$$

The concept of local Lipschitzness of set-valued maps will also be required to study strong invariance.

Definition 4. *A set-valued map $F : \mathcal{D} \subseteq \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is locally Lipschitz on the domain $\mathcal{D}$ if every point $x \in \mathcal{D}$ admits a neighborhood $U = U(x)$ and a positive constant $L = L(x)$ such that*

$$x_1, x_2 \in U \implies F(x_2) \subseteq F(x_1) + L \|x_1 - x_2\| B(0, 1). \tag{8}$$

We point out that $F(x)$ being locally Lipschitz implies that $F(x)$ is upper semicontinuous [14]. Using the preceding definitions, the following fundamental theorem describes how strong invariance of a set can be achieved with respect to a system described by a differential inclusion.

Theorem 1 (Adapted from [11])**.** *Let F be locally Lipschitz and suppose that F satisfies the Standing Hypotheses (Assumption 4). Then the following are equivalent for (6):*

(1) $F(x) \subseteq T_S(x)\ \forall x \in S$;
(2) (S, F) is strongly invariant.

Satisfying the conditions of Theorem 1 will play a central role in the analysis of the main results of this paper.

IV. Main Results

In this section we demonstrate how the conditions of Theorem 1 can be satisfied by design through solving a feasibility problem. We approach this problem by designing a differential inclusion of the form

$$G(x) = \{f(x) + g(x)u : u \in K(x)\} \tag{9}$$

where the set-valued map $K : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^m)$ satisfies $K(x) \subseteq \mathcal{U}$ for all $x \in \mathbb{R}^n$. The existence and behavior of solutions to the system dynamics (1) under any Lebesgue measurable $u \in K(x)$ can then be studied by analyzing $G(x)$ in light of the Standing Hypotheses and Theorem 1. This is a common method in the literature for considering all trajectories of a controlled system simultaneously [20, Ch. 3, §15], [11, Eq. (1.2)], [12, Ch. 10], [14, Eq. (34)].

A. Invariance of a Single Set

For simplicity of presentation, the first portion of our results consider a system with only one set S to be rendered forward invariant, defined as

$$S = \{x : h(x) \leq 0\}, \tag{10}$$

where h is a strict CBF satisfying Assumption 2. Considering multiple sets is analyzed in Section IV-B. We begin by defining the set-valued map K. In the prior work (see e.g., [4]), the forward invariance of a single set was guaranteed by considering a locally Lipschitz continuous control input u within the set

$$\{u \in \mathcal{U} : L_f h(x) + L_g h(x) u \leq -\alpha(h(x))\}, \tag{11}$$

for all $t \geq 0$. Inspired by this method, consider the set-valued map

$$K(x) = \left\{ u \in \mathbb{R}^m : \begin{bmatrix} A_S(x) \\ A_u \end{bmatrix} u \leq \begin{bmatrix} b_S(x) \\ b_u \end{bmatrix} \right\}, \tag{12}$$

where $A_S : \mathbb{R}^n \to \mathbb{R}^{q \times m}$ and $b_S : \mathbb{R}^n \to \mathbb{R}^q$ are defined as

$$A_S(x) = L_g h(x), \quad b_S(x) = -\alpha(h(x)) - L_f h(x). \tag{13}$$

Here, α is an extended class-$\mathcal{K}_\infty$ function which is locally Lipschitz on $\mathbb{R}$. Note that A_S and b_S are locally Lipschitz on $\mathbb{R}^n$. This holds since the functions f, g, $\frac{\partial h}{\partial x}$ are locally Lipschitz on $\mathbb{R}^n$. The set $K(x)$ can be considered as the *feasible set* of the combined set invariance and control input constraints for any given $x \in S$.

We define the closed set $\Omega \subset \mathbb{R}^n$ as having all of the following properties:

$$\begin{aligned} &\Omega \subset \{x \in \mathbb{R}^n : \text{int}(K(x)) \neq 0\}, \\ &\text{int}(\Omega) \neq \emptyset, \\ &\partial S \subset \text{int}(\Omega). \end{aligned} \tag{14}$$

Note that under Assumption 2, it holds that such an Ω exists and $\text{int}(S \cap \Omega) \neq \emptyset$. The following result presents conditions under which the interior of K is a locally Lipschitz set-valued map.

Lemma 1. *Let D be any bounded, open subset of Ω. Let K be defined as in (12). If A_S, b_S are locally Lipschitz on D, then $\text{int}(K)$ is locally Lipschitz continuous on D.*

The previous Lemma demonstrated that the function $\text{int}(K)$ is locally Lipschitz on any bounded, open subset $D \subset \Omega$. However, the Standing Hypotheses require a set-valued map $G(\cdot)$ which is *compact* for all values in its domain. To construct such a set-valued map, consider a bounded, open domain $D \subset \Omega$ and let $0 < \gamma < \inf_{x\in D} R_C(K(x))$, where

$$R_C(S) \triangleq \sup_{u\in S} d_{S^c}(u), \;\; S^c = \mathbb{R}^m \backslash S, \tag{15}$$

is the radius of the largest ball which can be inscribed in $S \subset \mathbb{R}^m$ [21, Sec 8.5]. We define the γ contraction of $K(x)$ as

$$\begin{aligned} K_\gamma(x) &= \text{int}(K(x)) - \gamma B(0,1), \qquad (16)\\ &= \{u \in K(x) : d_{K^c}(u) \geq \gamma\}, \;\; K^c = \mathbb{R}^m \backslash K(x), \end{aligned}$$

where $\text{int}(K(x)) - \gamma B(0,1)$ denotes the Minkowski difference. Note that $K_\gamma(x)$ is closed for all x in $D \subset \Omega$. The choice of parameter γ guarantees that $K_\gamma(x)$ is nonempty for all x in any bounded, open $D \subset \Omega$. Our next result shows that this set-valued map is locally Lipschitz continuous.

Lemma 2. *Let D be any bounded, open subset of Ω. If $\text{int}(K)$ is locally Lipschitz on a bounded, open set $D \subset \Omega$, then for any $0 < \gamma < \inf_{x\in D} R_C(x)$ it holds that K_γ is locally Lipschitz on D.*

Proof. Since $\text{int}(K)$ is nonempty and locally Lipschitz on D, then for all $x \in D$ there exists a domain $U(x)$ and constant $L = L(x)$ such that $\text{int}(K(y)) \subseteq \text{int}(K(z)) + L\|y - z\| B(0,1)$. Taking a Minkowski difference from both sides yields

$$\begin{aligned} \text{int}(K(y)) - \gamma B(0,1) \subseteq \text{int}(K(z)) - \gamma B(0,1) +& \\ L\|y-z\| B(0,1)& \\ \implies K_\gamma(y) \subseteq K_\gamma(z) + L\|y - z\| B(0,1).& \end{aligned}$$

The result follows. ∎

Note that since $R_C(x) > \gamma$ for all $x \in D \subset \Omega$, $K_\gamma(x)$ is never empty in D. The set-valued map $K_\gamma(\cdot)$ can be used to construct the following set-valued map $G_\gamma : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$:

$$G_\gamma(x) = \{v \in \mathbb{R}^n : v = f(x) + g(x)u, \;\; u \in K_\gamma(x)\}. \tag{17}$$

We next prove several properties about the set-valued map G_γ in order to show that it satisfies the conditions of Theorem 1. First, we show that G_γ is locally Lipschitz on any open, bounded subset of Ω.

Theorem 2. *Let D be any bounded, open subset of Ω. Then,*

i) the set-valued map G_γ from (17) is locally Lipschitz on D;
ii) the set-valued map G_γ from (17) satisfies the Standing Hypotheses from Assumption 4 for all x in D;
iii) the set-valued map G_γ from (17) satisfies $G_\gamma(x) \subseteq T_S(x)$ for all $x \in \mathbb{R}^n$.

Using Theorem 2, we can now prove the first main result of the paper that concerns the invariance of the set S for the closed-loop trajectories of (1).

Theorem 3. *Consider the system*

$$\dot{x}(t) \in G_\gamma(x(t)). \tag{18}$$

Let S be a set defined as in (10). Let $x(\cdot)$ be any trajectory of (18) under a Lebesgue measurable control input $u(\cdot)$ with $x_0 = x(0) \in \text{int}(S \cap \Omega)$. Let $[0, T(x_0))$ be the (possibly empty) maximal interval such that $x(t) \in \text{int}(\Omega)$ for all $t \in [0, T(x_0))$. Then $x(t) \in S$ for all $t \in [0, T(x_0))$.

Proof. Recall that Assumption 2 implies that $\partial S \subset \Omega$ and that $\text{int}(S \cap \Omega) \neq \emptyset$. By Theorem 2, G_γ satisfies the Standing Hypotheses from Definition 4 and is locally Lipschitz on $\text{int}(S \cap \Omega)$, which guarantees existence of solutions to (18). In addition, $G_\gamma(x) \subseteq T_S(x)$ for all $x \in \mathbb{R}^n$. Therefore by Theorem 1 the trajectory $x(t)$ will remain in S as long as $x(t) \in \text{int}(\Omega)$, implying $x(t) \in S$ for all $t \in [0, T(x_0))$. ∎

Theorem 3 considers the general case where $(S \setminus \Omega) \neq \emptyset$; i.e. there may exist interior points of S which are not in Ω.[2] Since the set-valued mapping G_γ satisfies the conditions of Theorem 1 only on bounded, open subsets of Ω, strong invariance cannot be guaranteed by Theorem 1 for any trajectory which leaves Ω. However in the case that $S \subseteq \Omega$, i.e. the interior of $K(x)$ is non-empty for all $x \in S$, the following corollary shows that the system pair (S, G) is strongly invariant.

Corollary 1. *Under the hypotheses of Theorem 3, suppose there exists a bounded, open domain $D \subseteq \Omega$ such that $S \subset D$. Then any Lebesgue measurable control input $u \in K_\gamma(x)$ renders the pair (S, G_γ) strongly invariant.*

B. Invariance of Multiple Sets

In this section, we discuss how to incorporate multiple safety requirements in an optimization framework and discuss conditions under which the resulting optimization problem is feasible. Consider the set of functions $h_i : \mathbb{R}^n \to \mathbb{R}$ defining the sets $S_i = \{x \mid h_i(x) \leq 0\}$, for $i = 1, 2, \ldots, N_h$. Defining the composed set $S_I = \bigcap_{i=1}^{N_h} S_i$, we seek to render the set S_I strongly invariant. Recall that the functions h_i satisfy Assumption 2. Incorporating general nonsmooth h_i functions into this analysis will be considered in future work.

[2] Recall that under Assumption 2 it holds that $\partial S \subset \Omega$ and $\text{int}(S \cap \Omega) \neq \emptyset$.

Similar to the previous section, we define the set-valued map $\widehat{K}(x)$ as

$$\widehat{K}(x) = \left\{ u \in \mathbb{R}^m : \begin{bmatrix} \hat{A}_S(x) \\ A_u \end{bmatrix} u \leq \begin{bmatrix} \hat{b}_S(x) \\ b_u \end{bmatrix} \right\}, \tag{19}$$

where we have

$$\hat{A}_S(x) = \begin{bmatrix} L_g h_1(x) \\ \vdots \\ L_g h_{N_h}(x) \end{bmatrix},$$
$$\hat{b}_S(x) = \begin{bmatrix} -\alpha_1(h_1(x)) - L_f h_1(x) \\ \vdots \\ -\alpha_{N_h}(h_{N_h}(x)) - L_f h_{N_h}(x) \end{bmatrix}. \tag{20}$$

Each $\alpha_i(\cdot)$ is a extended class-$\mathcal{K}_\infty$ function which is locally Lipschitz on $\mathbb{R}$. Using $\widehat{K}(x)$ we define the closed set $\widehat{\Omega} \subset \mathbb{R}^n$ as having all of the following properties:

$$\begin{aligned} &\widehat{\Omega} \subset \{x \in \mathbb{R}^n : \text{int}(\widehat{K}(x)) \neq 0\}, \\ &\text{int}(\widehat{\Omega}) \neq \emptyset, \\ &\partial S_i \subset \text{int}(\widehat{\Omega}) \ \forall i \in \{1, \dots, N_h\}. \end{aligned} \tag{21}$$

Given a bounded, open domain $D \subset \widehat{\Omega}$, we also define $\widehat{K}_{\hat{\gamma}}(x)$ as $\widehat{K}_{\hat{\gamma}}(x) = \text{int}(\widehat{K}(x)) - \hat{\gamma}B(0,1)$, where $\hat{\gamma}$ satisfies $0 < \hat{\gamma} < \inf_{x \in D} R_C(\widehat{K}(x))$ and $R_C(\cdot)$ is defined in (15). The set-valued map $\widehat{G}_{\hat{\gamma}}(x)$ is defined as

$$\widehat{G}_{\hat{\gamma}}(x) = \left\{ v \in \mathbb{R}^n : v = f(x) + g(x)u, \ u \in \widehat{K}_{\hat{\gamma}}(x) \right\}.$$

Using prior results, we can then show the following properties on $\widehat{G}_{\hat{\gamma}}(x)$.

Lemma 3. *Let $D \subset \widehat{\Omega}$ be a bounded, open set. Then the set-valued map $\widehat{G}_{\hat{\gamma}}$ is locally Lipschitz on D and satisfies all the Standing Hypotheses from Assumption 4 for all $x \in D$. Furthermore, for all $x \in D$ we have*

$$\widehat{G}_{\hat{\gamma}}(x) \subseteq \bigcap_{i=1}^{N_h} T_{S_i}(x) \tag{22}$$

We are ultimately interested in rendering the composed set S_I invariant by guaranteeing that $\widehat{G}_{\hat{\gamma}}(x) \subseteq T_{S_I}(x)$ for all $x \in S_I$. Note that in general S_I may have a boundary that cannot be described by a $\mathcal{C}^1$ function. From (22) the question remains as to whether $\bigcap_{i=1}^{N_h} T_{S_i}(x) \subseteq T_{S_I}(x)$ for all $x \in S_I$. If so, then by (22) it holds that $\widehat{G}_{\hat{\gamma}}(x) \subseteq T_{S_I}(x)$. Towards this end we review some mathematical preliminaries required to establish this condition. The concept of *transversality* was explored in [13] precisely to relate the intersection of tangent cones and the tangent cone of intersections of sets. Let $N_S(x)$ denote the normal cone of set S at x. Note that since each $h_i \in \mathcal{C}^{1,1}$, by Lemma 8 in [16, Appendix] it holds that $N_{S_i}(x) = N^P_{S_i}(x)$ for all $x \in S_i$, for all $i = 1, \dots, N_h$.

Definition 5. *Transversality holds for the pair (S_1, S_2) of two closed sets $S_1, S_2 \subset \mathbb{R}^n$ if for all $x \in \partial S_1 \cap \partial S_2$ we have the following:*

$$N_{S_1}(x) \cap (-N_{S_2}(x)) = \{0\}, \tag{23}$$

Again, from (22) we are interested in proving that $\bigcap_{i=1}^{N_h} T_{S_i}(x) \subseteq T_{S_I}(x)$ for all $x \in S_I$.

The result in [13, pp 99, 9.11 (d)] states the relation between the tangent and the normal cones of the intersection of two sets S_1, S_2, defined as zero sub-level sets of smooth functions, when the transversality condition holds. In short, if the transversality condition holds for the pair (S_1, S_2), then $T_{S_1 \cap S_2}(x) = T_{S_1}(x) \cap T_{S_2}(x)$. We can extend this result for arbitrary number of sets when pairwise transversality condition holds. Let $I : \mathbb{R}^n \to 2^{N_h}$ be the collection of indices of sets intersecting on their boundaries, defined as $I(x) = \{i \mid \exists j \neq i, h_i(x) = h_j(x) = 0\}$.

Lemma 4. *If the transversality condition holds for the pair (S_i, S_j) for all $i, j \in I(x)$, then*

$$T_{(\cap_i S_i)}(x) = \bigcap_{i=1}^{N_h} T_{S_i}(x), \tag{24}$$

holds for all $x \in (\bigcap_{i=1}^{N_h} S_i)$.

With the prior results, we are ready to present the following results on the strong invariance of S_I. Theorem 4 and Corollary 2, presented below, are the multiple-set counterparts of Theorem 3 and Corollary 1.

Theorem 4. *Consider the system*

$$\dot{x}(t) \in \widehat{G}_{\hat{\gamma}}(x(t)). \tag{25}$$

Consider the set $S_I = \bigcap_{i=1}^{N_h} S_i$ and suppose that the transversality condition holds for the pair (S_i, S_j) for all $i, j \in I(x)$. Let $x(\cdot)$ be any trajectory of (25) under a Lebesgue measurable control input $u(\cdot)$ with $x_0 = x(0) \in \text{int}(S_I \cap \widehat{\Omega})$. Let $[0, T(x_0))$ be the (possibly empty) maximal interval such that $x(t) \in \text{int}(\widehat{\Omega})$ for all $t \in [0, T(x_0))$. Then $x(t) \in S_I$ for all $t \in [0, T(x_0))$.

Proof. Since transversality holds for all $x \in S_I$, by Lemma 4 we have that $T_{S_I}(x) = \bigcap_{i=1}^{N_h} T_{S_i}(x)$ for all $x \in S_I$. The result then follows from Lemma 3 using similar arguments as in Theorem 3. ■

Similar to Corollary 1, we can state the following result for the case when $S_I \subset D$.

Corollary 2. *Under the hypotheses of Theorem 3, suppose there exists a bounded, open domain $D \subseteq \widehat{\Omega}$ such that $S_I \subset D$. If $x_0 = x(0) \in S_I$ then any Lebesgue measurable control input $u(t) \in \widehat{K}_{\hat{\gamma}}(x(t))$ renders the pair $(S_I, \widehat{G}_{\hat{\gamma}})$ strongly invariant; i.e. $x(t) \in S_I$ for all $t \geq 0$.*

We now present an optimization problem which generates control inputs $u(t)$ which lie in the interior of $\widehat{K}(x)$. Define $z = [\, v^T \ \delta_1 \ \dots \ \delta_{N_h} \,]^T$ and consider the optimization problem

$$\min_{v, \delta_1, \delta_2, \dots, \delta_{N_h}} \quad C(z) \tag{26a}$$
$$\text{s.t.} \quad A_u v \leq b_u, \tag{26b}$$
$$L_{f_i} h_{s_i} + L_{g_i} h_{s_i} v \leq -\delta_i h_i, \quad i = 1, 2, \dots, N_h \tag{26c}$$

where $C : \mathbb{R}^{m+N_h} : \mathbb{R}$ is a convex objective function. The form of $C(\cdot)$ determines the classification of the optimization problem in (26); for example $C(\cdot)$ can be chosen as $C(z) = 0$ to make (26) a simple feasibility problem, as $C(z) = F^T z$ for some $F \in \mathbb{R}^{N_h+m}$ to make (26) a linear program, or as $C(z) = \frac{1}{2} z^T Q z + F^T z$ for some positive definite $Q \in \mathbb{R}^{(N_h+m)\times(N_h+m)}$ to make (26) a quadratic program. In (26) the variable δ_i is the slack variable corresponding to the i-th safety constraint, which helps guarantee the feasibility of the optimization problem as discussed in the next Theorem. Under the conditions of this paper, any Lebesgue-measurable $u(t)$ computed from (26) will render the set S_I invariant as per the results in Theorem 4 and Corollary 2. The following final main result of this paper provides guarantees on the feasibility of the optimization problem in (26).

Theorem 5. *Suppose that the transversality condition holds for pair of any two sets (S_i, S_j) for all $i, j \in I(x)$, and $x \in \partial S_i \cap \partial S_j$. Then under Assumption 2, the optimization problem (26) is feasible for all $x \in S_I$, and the set $\widehat{K}(x)$ from (19) has a non-empty interior for all $x \in S_I$.*

Proof. Let $x \in \text{int}(\bigcap S_i)$. Then, we have that $h_i(x) \neq 0$ for all $i = 1, 2, \ldots, N_h$. Choose any $\bar{v} \in \mathcal{U}$ so that (26b) holds. Then, with this $\bar{v}$, define $\bar{\delta}_i = \frac{L_{f_i} h_{s_i} + L_{g_i} h_{s_i} v_i}{h_i}$, which is well-defined for all $x \in \text{int}(\bigcap S_i)$. Thus, we have that there exists a solution such that (26b)-(26c) holds, implying the optimization problem (26) is feasible for all $x \in \text{int}(\bigcap S_i)$. Now, with $\bar{\delta}_i$ defined as above choose any $\hat{\delta}_i > \max\{0, \bar{\sup}_x |\delta_i(x)|\}$ so that $\hat{\delta}_i - \bar{\delta}_i > 0$ for all $i \in 1, 2, \ldots, N_h$. Then, with this choice of $\hat{\delta}_i$ we have that $L_{f_i} h_{s_i} + L_{g_i} h_{s_i} \bar{v}_i + \hat{\delta}_i h_i < L_{f_i} h_{s_i} + L_{g_i} h_{s_i} \bar{v}_i + \bar{\delta}_i h_i \leq 0$, which implies that $(\bar{v}_i, \hat{\delta}_i)$ satisfy (26c) with strict inequality. Also, note that $\alpha_i(h_i) = \hat{\delta}_i h_i$ is an extended class $\mathcal{K}_\infty$ function for each $i = 1, 2, \ldots, N_h$, since $\hat{\delta}_i > 0$. Thus, we have that there exists $u \in \mathcal{U}$ such that $L_f h_i + L_g h_i u < -\alpha_i(h_i)$, for any $x \in \text{int}(\bigcap S_i)$, and hence, $\widehat{K}(x)$ has a non-empty interior in that domain. Next, we show that for any $x \in \bigcap \partial S_i$, the set $\widehat{K}$ has a non-empty interior. Under Assumption 2, there exists $u \in \mathcal{U}$ such that the inequalities (26c) strictly hold for all $i \in I(x)$ for all $x \in \bigcap \partial S_i$. For any $j \notin I(x)$, we have that $h_j(x) \neq 0$, and the analysis above guarantees there exists a strict solution for (26c) for $j \notin I(x)$. Thus, there exists a strict solution of (26c) for all $x \in \bigcap S_i$, this, $\text{int}(\widehat{K}(x))$ is non-empty for all $x \in \bigcap S_i$. ■

Theorem 5 demonstrates that we can guarantee forward-invariance of multiple safe sets by solving the optimization problem (26) with additional non-negative slack variables in (26c). By Theorem 5 the optimization problem is guaranteed to be feasible at all points $x \in \bigcap_{i=1}^{N_h} S_i$, and the non-emptiness of the set-valued map $\text{int}(\widehat{K}(x))$ is also guaranteed for all $x \in \bigcap_{i=1}^{N_h} S_i$, which in turn guarantees forward-invariance of the multiple safe sets.

V. Conclusion

This paper presented a method to guarantee the forward invariance of composed sets using control barrier functions and incorporating input constraints. We demonstrated that control inputs rendering these sets invariant can be computed by solving a feasibility optimization problem. The computed control inputs are only required to be Lebesgue measurable and need not be continuous. Future work will incorporate more general control constraints and nonsmooth control barrier functions.

References

[1] M. Z. Romdlony and B. Jayawardhana, "Stabilization with guaranteed safety using control Lyapunov-barrier function," *Automatica*, vol. 66, pp. 39–47, 2016.

[2] A. D. Ames, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs with application to adaptive cruise control," in *53rd Conference on Decision and Control*. IEEE, 2014, pp. 6271–6278.

[3] A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, "Control barrier functions: Theory and applications," in *2019 18th European Control Conference*. IEEE, pp. 3420–3431.

[4] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3861–3876, 2017.

[5] A. Li, L. Wang, P. Pierpaoli, and M. Egerstedt, "Formally correct composition of coordinated behaviors using control barrier certificates," in *IEEE/RSJ International Conference on Intelligent Robots and Systems*. IEEE, 2018, pp. 3723–3729.

[6] P. Glotfelter, J. Cortés, and M. Egerstedt, "Nonsmooth barrier functions with applications to multi-robot systems," *IEEE control systems letters*, vol. 1, no. 2, pp. 310–315, 2017.

[7] ——, "Boolean composability of constraints and control synthesis for multi-robot systems via nonsmooth control barrier functions," in *2018 Conference on Control Technology and Applications*. IEEE, pp. 897–902.

[8] A. V. Fiacco, "Sensitivity analysis for nonlinear programming using penalty methods," *Mathematical Programming*, vol. 10, no. 1, pp. 287–311, 1976.

[9] S. M. Robinson, "Perturbed Kuhn-Tucker points and rates of convergence for a class of nonlinear-programming algorithms," *Mathematical Programming*, vol. 7, no. 1, pp. 1–16, 1974.

[10] X. Xu, P. Tabuada, J. W. Grizzle, and A. D. Ames, "Robustness of control barrier functions for safety critical control," *IFAC-PapersOnLine*, vol. 48, no. 27, pp. 54–61, 2015.

[11] P. Clarke, Y. S. Ledyaev, R. Stern, and P. Wolenski, "Qualitative properties of trajectories of control systems: a survey," *Journal of dynamical and control systems*, vol. 1, no. 1, pp. 1–48, 1995.

[12] J.-P. Aubin and H. Frankowska, *Set-valued analysis*. Springer Science & Business Media, 2009.

[13] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, and P. R. Wolenski, *Nonsmooth analysis and control theory*. Springer Science & Business Media, 2008, vol. 178.

[14] J. Cortes, "Discontinuous dynamical systems," *IEEE Control systems magazine*, vol. 28, no. 3, pp. 36–73, 2008.

[15] Y. Emam, P. Glotfelter, and M. Egerstedt, "Robust barrier functions for a fully autonomous, remotely accessible swarm-robotics testbed," *arXiv preprint arXiv:1909.02966*, 2019.

[16] J. Usevitch, K. Garg, and D. Panagou, "Strong invariance using control barrier functions: A clarke tangent cone approach," *arXiv preprint arXiv:2004.03733*, 2020.

[17] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3861–3876, 2016.

[18] W. S. Cortez, D. Oetomo, C. Manzie, and P. Choong, "Control barrier functions for mechanical systems: Theory and application to robotic grasping," *IEEE Transactions on Control Systems Technology*, 2019.

[19] A. D. Ames, K. Galloway, K. Sreenath, and J. W. Grizzle, "Rapidly exponentially stabilizing control Lyapunov functions and hybrid zero dynamics," *IEEE Transactions on Automatic Control*, vol. 59, no. 4, pp. 876–891, 2014.

[20] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides: Control Systems*. Springer Science & Business Media, 2013, vol. 18.

[21] S. Boyd, S. P. Boyd, and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.